ASSOCIATED GRADED MODULES OVER NOETHERIAN LOCAL RINGS

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1. Introduction

Let $T = \mathsf{k}[[x_1, \ldots, x_n]]$, $S = G_{\mathfrak{m}}(T) = \mathsf{k}[x_1, \ldots, x_n]$ and M be a T-module. In [9], given the hypothesis that $G_{\mathfrak{m}}(M)$ has a pure resolution, Puthenpurakal constructs a pure S-resolution of $G_{\mathfrak{m}}(M)$ from a T-resolution of M. Moreover, if M is Cohen-Macaulay, he also gives a characterization for $G_{\mathfrak{m}}(M)$ to have a pure resolution. This article is motivated by questions on the generalization of these results to modules over Noetherian local rings.

In this paper, we study the properties of modules with finite projective dimension over a Noetherian local ring R, whose associated graded modules have pure resolutions. In [11], Sammartano proved that the Betti numbers of M can be obtained from those of $G_{\mathfrak{m}}(M)$ by negative consecutive cancellations. In particular, if $G_{\mathfrak{m}}(M)$ has a pure resolution, the Betti numbers of $G_{\mathfrak{m}}(M)$ are equal to the Betti numbers of M. Given a free resolution F of an R-module M, we construct a complex F^* of free modules over the associated graded ring A of R. This construction is crucial in proving the major results of this article. In particular, we prove that $G_{\mathfrak{m}}(M)$ has a pure A-resolution if and only if F^* is a resolution of $G_{\mathfrak{m}}(M)$. As a consequence, we obtain that the existence of an R-module M with certain properties implies that R is Cohen-Macaulay (in the same vein as Theorem 4.9 of [1]). We also prove a local version of Herzog and Kühl's celebrated result ([5]), which was further generalized in [2] and [1].

This paper is organized as follows. In section 2, we introduce the notation, definitions, basic observations, and previous results that are needed in the rest of the article.

Section 3 is devoted to the study of properties of N^* (see Definition 2.8(c)), where N is the submodule of a free R-module F. The key result of this section involves a characterization of N^* being an equigenerated graded A-module (Proposition 3.6).

Section 4 provides the culmination of the theory introduced in sections 2 and 3. We construct a complex $\mathbb{F}^*_\bullet$ from a resolution $\mathbb{F}_\bullet$ of M and use Proposition 3.6 to prove that it is a free resolution of $G_{\mathfrak{m}}(M)$ under certain conditions (Theorem 4.5). We also give sufficient conditions for R to be Cohen-Macaulay (Theorem 4.8) and prove the local version of Herzog and Kühl's result (Theorem 4.12) promised earlier.

2. Preliminaries

2.1. Graded Betti Numbers and Pure Resolutions.

Definition 2.1. a) *Let R be a ring. We say that R is* graded *if there exists a decomposition (as abelian groups) $R = \bigoplus_{i\in\mathbb{Z}} R_i$ such that $R_iR_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$.*

b) *A graded ring R is said to be* nonnegatively graded *if $R_i = 0$ for all $i < 0$.*

c) *A nonnegatively graded ring R is said to be* standard graded *if $R = R_0[R_1]$ as a R_0-algebra, where $R_0 = \mathsf{k}$, a field.*

d) *Let M be a module over a graded ring R. We say that M is a* graded R-module *if there exists a decomposition (as abelian groups) $M = \bigoplus_{i\in\mathbb{Z}} M_i$ such that $R_iM_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$.*

e) *The n-*twist *of a graded module M, denoted by $M(n)$, is the graded module defined as $M(n)_i = M_{n+i}$ for all $i \in \mathbb{Z}$.*

f) *Let M, N be graded R-modules. Then an R-linear map $\phi : M \to N$ called a* graded map of degree n *if $\phi(M_i) \subset N_{n+i}$ for all $i \in \mathbb{Z}$. By convention, the term 'graded map' means a graded map of degree zero.*

Definition 2.2. *Let M be a graded R-module and*

$$\mathbb{F}_\bullet : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

be a free resolution of M.

a) *If each ϕ_i is a graded map of degree zero, then we say that $\mathbb{F}_\bullet$ is a* graded free resolution *of M.*
b) *We say that the resolution $\mathbb{F}_\bullet$ is* minimal *if $\phi_i(F_i) \subset \mathfrak{m}F_{i-1}$ for all $i \geq 1$.*
c) *If $\mathbb{F}_\bullet$ is a graded minimal free resolution of M, then the module $\Omega_i^R(M) = \ker(\phi_{i-1})$ is a graded R-module, called the i^{th}* syzygy module *of M with respect to the resolution $\mathbb{F}_\bullet$. The number of minimal generators of $\Omega_i^R(M)$ in degree j is denoted by $\beta_{i,j}(M)$, and is called the $(i,j)^{th}$* graded Betti number *of M. The number $\beta_i(M) = \sum_j \beta_{i,j}(M)$ is called the* total i^{th} Betti number of M, *and it is the number of elements in a minimal generating set of $\Omega_i(M)$.*
d) *The series $\mathcal{P}_M^R(z) = \sum_{i\geq 0} \beta_i(M) z^i$ (or simply $\mathcal{P}_M(z)$) is called as the* Poincare series of M, *and the series $\mathcal{P}_M^R(s,t) = \sum_{i,j} \beta_{i,j}(M) s^i t^j$ (or simply $\mathcal{P}_M(s,t)$) is called as the* graded Poincare series *of M.*
e) *The* projective dimension of M, *denoted by $\operatorname{pdim}_R(M)$ or simply by $\operatorname{pdim}(M)$, is the length of a minimal graded free resolution of M.*
f) *The* regularity of M *as*
$$\operatorname{reg}(M) = \sup\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$
g) *The resolution $\mathbb{F}_\bullet$ is said to be* pure *if for every i, $\beta_{i,j}(M) \neq 0$ for at most one j. A module with a pure resolution is called as a* pure module.
h) *A pure resolution $\mathbb{F}_\bullet$ of a module M generated in degree 0 is said to be* linear *if $\beta_{i,j} \neq 0$ implies $j = i$.*
i) *A pure R-module M is said to be of type*
   1) *$\delta = (\delta_0, \delta_1, \delta_2, \ldots)$ if $\operatorname{pdim}(M) = \infty$ and $\beta_{i,\delta_i}(M) \neq 0$ for all $i \geq 0$.*
   2) *$\delta = (\delta_0, \delta_1, \ldots, \delta_p, \infty, \infty, \ldots)$ if $\operatorname{pdim}(M) = p$ and $\beta_{i,\delta_i}(M) \neq 0$ for $0 \leq i \leq p$.*

Definition 2.3. *Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded module over a k-algebra R. Then the function $H_M : \mathbb{Z} \to \mathbb{Z}$, given by $H_M(n) = \dim_{\mathsf{k}}(M_n)$ is called as* Hilbert function of M, *and the series $H_M(z) = \sum_{n\in\mathbb{Z}} H_M(n) z^n$ is called as the* Hilbert series of M.

Remark 2.4. It is well known that (e.g., see [**?**, Section 4.1]) if M is a finitely generated R-module, then

a) There exists a polynomial $P(x) \in \mathbb{Q}[x]$ such that $H_M(n) = P(n)$ for $n \gg 0$.
b) There exists $f(z) \in \mathbb{Z}[z, z^{-1}]$ such that $H_M(z) = f(z)/(1-z)^d$, where $d = \dim(M)$ and $f(1) \neq 0$.

Definition 2.5. *Let M be a graded R-module of dimension d and $H_M(z) = f(z)/(1-z)^d$. Then the number $f(1)$ is called as the multiplicity of M, and we denote it by $e(M)$.*

2.2. **Cohen-Macaulay Defect.**

Definition 2.6. *Let A be a standard graded k-algebra, and M be a finitely generated graded A-module. The Cohen-Macaulay defect of M, denoted $\operatorname{cmd}(M)$, is defined as $\operatorname{cmd}(M) = \dim(M) - \operatorname{depth}(M)$.*

We record some observations and known results related to Cohen-Macaulay defect in the following remark.

Remark 2.7. Let A and M be as above.

a) M is Cohen-Macaulay if and only if $\operatorname{cmd}(M) = 0$.
b) If $\operatorname{pdim}_A(M) < \infty$, then $\operatorname{cmd}(M) = \operatorname{cmd}(A)$ if and only if $\operatorname{codim}(M) = \operatorname{pdim}_A(M)$.
c) ([1, Proposition 3.7]) If M is pure, then $\operatorname{codim}(M) \leq \operatorname{pdim}_A(M)$.
   Moreover, if $\operatorname{pdim}_A(M) < \infty$, then $\operatorname{cmd}(A) \leq \operatorname{cmd}(M)$.
d) ([1, Theorem 3.9]) Let M be a pure A-module of type $\delta = (\delta_0, \ldots, \delta_p)$, and $b_i = (-1)^{i-1} \prod_{j\neq i} \frac{\delta_j - \delta_0}{\delta_j - \delta_i}$ for $i = 1, \ldots, p$. Then $\operatorname{cmd}(M) = \operatorname{cmd}(A)$ if and only if $\beta_i = b_i \beta_0$ for $i = 1, \ldots, p$.

Definition 2.6, and the observations (a) and (b) in the Remark 2.7 are also valid for finitely generated modules over a Noetherian local ring. In this case, results analogous to Remark 2.7(c) and (d) are also true, as proved in Theorems 4.8 and 4.11.

2.3. Associated Graded Rings and Modules.

Definition 2.8. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring, and M be a finitely generated R-module.*

a) *Then the ring*
$$A := G_{\mathfrak{m}}(R) = \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$
is called the associated graded ring *of R with respect to $\mathfrak{m}$, and the A module*
$$G_{\mathfrak{m}}(M) = \bigoplus_{i\geq 0} \mathfrak{m}^i M/\mathfrak{m}^{i+1}M$$
is called the associated graded module *of M with respect to $\mathfrak{m}$.*

b) *Given any nonzero element $x \in M$, define $\nu(x) = \min\{i \mid x \in \mathfrak{m}^i M \setminus \mathfrak{m}^{i+1}M\}$. If $\nu(x) = i$, we define an element of degree i in $G_{\mathfrak{m}}(M)$ naturally associated to x as follows: define $x^* = x + \mathfrak{m}^{i+1}M \in \mathfrak{m}^i M/\mathfrak{m}^{i+1}M \subset G_{\mathfrak{m}}(M)$.*

c) *Given a nonzero submodule N of M, we define $N^* = \langle x^* \in G_{\mathfrak{m}}(M) \mid x \in N\rangle$. Furthermore, we define order of N as $\nu(N) = \min\{\nu(x) \mid x \in N \setminus \{0\}\}$.*

d) *Let $\phi : R^m \to R^n$ be a non-zero R-linear map. Considering ϕ as a matrix in the free module R^{mn}, we define the* initial form *of ϕ, to be the corresponding matrix $\phi^* \in A^{mn}$. In other words, if $\phi = (a_{ij})$ and $\nu(\phi) = s$, then $\phi^* = (a_{ij} + \mathfrak{m}^{s+1})$.*

Remark 2.9. Using the representation in the Definition 2.8 (d), observe that

a) $\phi^* : G_{\mathfrak{m}}(R^m) \to G_{\mathfrak{m}}(R^n)$ is a graded map of degree s.
Notation: We also use ϕ^* to denote the induced map of degree zero from $G_{\mathfrak{m}}(R^m)(-s-j)$ to $G_{\mathfrak{m}}(R^n)(-j)$ for all $j \in \mathbb{Z}$.

b) If $\psi : R^k \to R^m$ is a non-zero R-linear map such that $\phi \circ \psi = 0$, then $\phi^* \circ \psi^* = 0$.

Remark 2.10. Let M be an R-module generated by $u_1, \dots, u_l$ and F be a free R-module with basis $\{w_1, \dots, w_l\}$. Then the map $\phi_0 : F \to M$ be defined as $\phi_0(w_i) = u_i$ induces a natural A-linear onto map $\epsilon : A^l \to G_{\mathfrak{m}}(M)$ defined as $\epsilon(w_i^*) = u_i^*$.
Furthermore, if $\{u_1, \dots, u_l\}$ is a minimal generating set of M, then by Nakayama lemma, $u_i \notin \mathfrak{m}M$. The A-module $G_{\mathfrak{m}}(M)$ is minimally generated by $\{u_1^*, u_2^*, \dots, u_l^*\} \subset M/\mathfrak{m}M$. In particular, $G_{\mathfrak{m}}(M)$ is generated in degree zero.

Lemma 2.11. *Let N be a submodule of a free R-module F, and $M = F/N$. Then, N^* is the kernel of the natural map $\epsilon : G_{\mathfrak{m}}(F) \to G_{\mathfrak{m}}(M)$.*

Proof. Observe that $(G_{\mathfrak{m}}(M))_i \simeq (\mathfrak{m}^i F + N)/(\mathfrak{m}^{i+1}F + N)$. Let $x \in N$, and $\nu(x) = s$ in F. Then, $x^* = x + \mathfrak{m}^{s+1}F$ and $\epsilon(x^*) = x + \mathfrak{m}^{s+1}F + N = 0$. Hence, $N^* \subset \ker(\epsilon)$.
Let $x + \mathfrak{m}^{s+1}F \in \ker(\epsilon) \setminus \{0\}$. Then, $x \in \mathfrak{m}^{s+1}F + N$. Let $x = y + z$, where $y \in \mathfrak{m}^{s+1}F$ and $z \in N$. Thus, $z \in \mathfrak{m}^s F \setminus \mathfrak{m}^{s+1}F$ and $x^* = z^*$. Hence, $\ker(\epsilon) \subset N^*$. □

Question 2.12. *Let N be a submodule of M. Suppose $\{v_1, \dots, v_k\}$ is a minimal generating set of N. Then $\langle v_1^*, \dots, v_k^*\rangle \subset N^*$. When does the equality hold?*

The following example shows that in general $N^* \not\subset \langle v_1^*, \dots, v_k^*\rangle$.

Example 2.13. *Let*
$$R = \mathsf{k}[[X, Y, Z]]/\langle XZ - Y^3, YZ - X^4, Z^2 - X^3Y^2\rangle.$$
Then
$$A = G_{\mathfrak{m}}(R) \simeq \mathsf{k}[x, y, z]/\langle xz, yz, z^2, y^4\rangle.$$
Here, for $N = \langle X\rangle$, we have $N^* = \langle x, y^3\rangle$. So, $N^* \neq \langle X^*\rangle$.

Definition 2.14. *A subset* $\{v_1, \dots, v_r\}$ *of an* R*-module* K *is said to be a* standard basis *of* K *if* $\{v_1^*, \dots, v_r^*\} = N^*$.
A standard basis of N *is said to be* minimal *if none of its proper subsets is a standard basis of* N.

Remark 2.15. Every standard basis of N forms a generating set of N (cf. [7, Proposition 2.1]).

3. Submodule of a Free Module and an Induced Filtration

Lemma 3.1. *Let* N *be a submodule of* F. *Let* $\mathcal{F} = \{N_i = \mathfrak{m}^i F \cap N\}_{i\in\mathbb{Z}}$, *and* $G_{\mathcal{F}}(N) = \bigoplus_{i\geq 0}(N \cap \mathfrak{m}^i F)/(N \cap \mathfrak{m}^{i+1}F)$. *Then we have* $N^* \simeq G_{\mathcal{F}}(N)$.

Proof. Since $\mathfrak{m}^{i+1}F \cap N = \mathfrak{m}^{i+1}F \cap (N \cap \mathfrak{m}^i F)$, we have the natural isomorphism of R-modules

$$(\mathfrak{m}^i F \cap N)/(\mathfrak{m}^{i+1}F \cap N) \simeq (\mathfrak{m}^i F \cap N + \mathfrak{m}^{i+1}F)/\mathfrak{m}^{i+1}F$$

for each $i \geq 0$. The above isomorphism, followed by the natural inclusion $(\mathfrak{m}^i F \cap N + \mathfrak{m}^{i+1}F)/\mathfrak{m}^{i+1}F \subset G_{\mathfrak{m}}(M)$ is an additive function given by $x + \mathfrak{m}^{i+1}F \cap N \mapsto x + \mathfrak{m}^{i+1}F$. This induces an additive function $\eta : G_{\mathcal{F}}(N) \to G_{\mathfrak{m}}(F)$ defined as

$$\eta\left(\sum_{i\geq 0} x_i + (\mathfrak{m}^{i+1}F \cap N)\right) = \sum_{i\geq 0} x_i + \mathfrak{m}^{i+1}F$$

where $x_i + \mathfrak{m}^{i+1}F \cap N \in (G_{\mathcal{F}}(N))_i$. We now prove that
(a) η is A-linear, (b) η is injective, (c) $\mathrm{Im}(\eta) = N^*$, which proves the lemma.
Let $a^* \in A$ be of degree j, and $\bar{x} = x + \mathfrak{m}^{i+1}F \cap N \in G_{\mathcal{F}}(N)$ be nonzero. Then $a^*\bar{x} = ax + \mathfrak{m}^{i+j+1}F \cap N$. It is clear that $\eta(a^*\bar{x}) = a^*\eta(\bar{x})$. This fact, together with the additivity of η, shows that η is A-linear.
Let $x + N \cap \mathfrak{m}^{i+1}F \in G_{\mathcal{F}}(N)$ be nonzero. Then $x \notin \mathfrak{m}^{i+1}F$, and hence $\eta(x + N \cap \mathfrak{m}^{i+1}F) = x + \mathfrak{m}^{i+1}F \neq 0$. Therefore, that η is injective.
Consider $x \in N$ and suppose that $\nu(x) = i$, i.e., $x \in \mathfrak{m}^i F \setminus \mathfrak{m}^{i+1}F$. Then $x^* = x + \mathfrak{m}^{i+1}F = \eta(x + N \cap \mathfrak{m}^{i+1}F)$. Therefore, $N^* \subset \mathrm{Im}(\eta)$.
To see the other inclusion, let $x + \mathfrak{m}^{i+i}F \in \mathrm{Im}(\eta)$ be a nonzero homogeneous element. Then $x + \mathfrak{m}^{i+1}F = \eta(y + \mathfrak{m}^{i+1}F \cap N)$ for some $y \in \mathfrak{m}^i F \cap N$. Since $x + \mathfrak{m}^{i+1}F$ is nonzero, we have $\nu(y) = i$. So, $y^* = y + \mathfrak{m}^{i+1}F = x + \mathfrak{m}^{i+1}F$. Hence, $\mathrm{Im}(\eta) \subset N^*$. □

Definition 3.2. *A* $\mathbb{Z}$*-graded finitely generated module* M *is said to be* equigenerated *if there exists* $n \in \mathbb{Z}$ *such that* $M = \langle v_1, \dots, v_r\rangle$ *with* $\deg(v_i) = n$ *for all* i.

Lemma 3.3. *Let* N *be a submodule of a finite rank free* R*-module* F *with* $\nu(N) = s$. *Let* $\mathcal{F}$ *and* $G_{\mathcal{F}}(N)$ *be as in Lemma 3.1. If* $G_{\mathcal{F}}(N)$ *is equigenerated, then* $N \cap \mathfrak{m}^i F = \mathfrak{m}^{i-s}N$ *for all* $i \geq s$.

Proof. Since $N^* = \langle v^* \mid v \in N\rangle$, and $\nu(v_i) \geq s$ for all i, we see that $\nu(v) \geq s$ for all $v \in N \setminus \{0\}$. Moreover, by hypothesis, $\nu(v_i) = s$ for some i. Hence, by Lemma 3.1, since $G_{\mathcal{F}}(N) \simeq N^*$, there is a minimal generator of $G_{\mathcal{F}}(N)$ in degree s. Thus, by hypothesis, it follows that $G_{\mathcal{F}}(N)$ is generated in degree s. So, we have $N_i = N$ for $i \leq s$ and

$$\frac{N_{s+j}}{N_{s+j+1}} = \mathfrak{m}^j \frac{N_s}{N_{s+1}} \Rightarrow N_{s+j} = \mathfrak{m}^j N_s + N_{s+j+1} = \mathfrak{m}^j N + N_{s+j+1},$$

for $j \geq 1$. By the Artin-Rees lemma, there exists j_0 such that $N_{s+j+1} = \mathfrak{m}N_{s+j}$ for all $j \geq j_0$. For $j \geq j_0$, $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1} = \mathfrak{m}^j N + \mathfrak{m}N_{s+j}$. By Nakayama Lemma, $N_{s+j} = \mathfrak{m}^j N$ for $j \geq j_0$.
We show by descending induction that $N_{s+j} = \mathfrak{m}^j N$ for all $j \leq j_0$. This is true for $j = j_0$ by the previous argument. Assume $N_{s+j+1} = \mathfrak{m}^{j+1}N$ for some $j \leq j_0 - 1$. Then,

$$\mathfrak{m}^{j+1}N \subset \mathfrak{m}N_{s+j} \subset N_{s+j+1} = \mathfrak{m}^{j+1}N.$$

Hence, $N_{s+j+1} = \mathfrak{m}N_{s+j}$ and $N_{s+j} = \mathfrak{m}^j N + \mathfrak{m}N_{s+j}$. By Nakayama Lemma, $N_{s+j} = \mathfrak{m}^j N$ for $j \leq j_0 - 1$. □

The converse of the above result is true. In fact, we prove a stronger statement.

Lemma 3.4. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and N be a submodule of a free module F. Let $\{v_1, \ldots, v_k\}$ be a minimal generating set of N with $\nu(v_j) \geq s$ for all j. Suppose that $N \cap \mathfrak{m}^i F = \mathfrak{m}^{i-s} N$ for some $i > s$, then no minimal generator of N^* has degree i, and $\nu(v_j) < i$ for all j.*

Proof. We show that $(N^*)_i \subset \mathfrak{n} N^*$. Note that $(N^*)_i = (\mathfrak{m}^i F \cap N)/(\mathfrak{m}^{i+1} F \cap N) = \mathfrak{m}^{i-s} N/(\mathfrak{m}^{i+1} F \cap N)$. Let $y \in (N^*)_i$. Then $y = x + \mathfrak{m}^{i+1} F \cap N$ for some $x \in \mathfrak{m}^{i-s} N$. So, $x = \sum_{j=1}^{k} a_j v_j$, where $a_j \in \mathfrak{m}^{i-s}$. Therefore,

$$y = x + \mathfrak{m}^{i+1} F \cap N = \sum_{j=1}^{k} (a_j + \mathfrak{m}^{i-s+1})(v_j + (\mathfrak{m}^{s+1} F \cap N)) \in \mathfrak{n} N^*.$$

This completes the proof. □

The special case of $i = s + 1$ in the previous lemma is interesting, which we record in the following:

Lemma 3.5. *Let N be a nonzero submodule of a finite rank free R-module F, and let $\{v_1, \ldots, v_k\}$ be a minimal generating set of N with $\nu(v_i) \geq s$ for all i. If $N \cap \mathfrak{m}^{s+1} F = \mathfrak{m} N$, then $\nu(v_i) = s$ for all i, and $\{v_1^*, \ldots, v_k^*\}$ is a part of a minimal generating set of N^*, and N^* does not have a minimal generator in degree $s + 1$.*
Furthermore, if $\mu(N^*) = k$, then $\{v_1, \ldots, v_k\}$ forms a standard basis for N.

Proof. Since no minimal generator of N can be in $\mathfrak{m} N$, the condition $N \cap \mathfrak{m}^{s+1} F = \mathfrak{m} N$ implies that $\nu(v_i) = s$ for all i.
Let $\mathfrak{n}$ denote the homogeneous maximal ideal of A. Suppose $\alpha_1, \ldots, \alpha_k \in A$ are such that $\sum_i \alpha_i v_i^* \in \mathfrak{n} N^*$. It suffices to show that $\alpha_i \in \mathfrak{n}$ for all i. If $\alpha_j \in \mathfrak{n}$ for some j, then $\alpha_j v_j^* \in \mathfrak{n} N^*$ and $\sum_{i \neq j} \alpha_i v_i^* \in \mathfrak{n} N^*$. So, suppose that $\alpha_i \notin \mathfrak{n}$ for all i. Hence, for each i, we have $\alpha_i = \sum_j a_{i,j} + \mathfrak{m}^{j+1}$ with $a_{i,0} \in R \backslash m$. Then, from $\sum_i \left(\sum_j a_{i,j} + \mathfrak{m}^{j+1} \right) v_i^* \in \mathfrak{n} N^*$ we get $\sum_i (a_{i,0} + \mathfrak{m}) v_i^* \in \mathfrak{n} N^*$.
Therefore, $\sum_i a_{i,0} v_i \in N \cap \mathfrak{m}^{s+1} F = \mathfrak{m} N$. Since $\{v_1, \ldots, v_k\}$ is a minimal generating set of N, we get that $a_{i,0} \in \mathfrak{m}$ for all i, which is a contradiction. Hence, we must have $\alpha_i \in \mathfrak{n}$ for all i. Hence, $\{v_1^*, \ldots, v_k^*\}$ can be extended to a minimal generating set of N^*. This completes the proof of the first part.
Moreover, if $\mu(N^*) = k$, then we see that $\{v_1^*, \ldots, v_k^*\}$ is a minimal generating set for N^*, i.e., $\{v_1, \ldots, v_k\}$ is a standard basis of N . □

Our goal is to study when $G_{\mathfrak{m}}(M)$ has a pure resolution. Lemma 3.1 tells us that $G_{\mathcal{F}}(N)$ must be equigenerated for $G_{\mathfrak{m}}(M)$ to have a pure resolution. We study this condition further in the next proposition. In particular, we get some positive answers to Question 2.12.

Proposition 3.6. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring, $A = G_{\mathfrak{m}}(R)$, and M be a finitely generated R-module. Consider the exact sequence $0 \to N \to F \to M \to 0$, where F is free R-module, and $\nu(N) = s$. Suppose $\{v_1, \ldots, v_k\}$ is a minimal generating set of N. Let $\mathcal{F}$ and $G_{\mathcal{F}}(N)$ be as in Lemma 3.1. Then the following statements are equivalent:*
(i) *$G_{\mathcal{F}}(N)$ is equigenerated.*
(ii) *$G_{\mathcal{F}}(N) \simeq G_{\mathfrak{m}}(N)(-s)$.*
(iii) *$N \cap \mathfrak{m}^i F = \mathfrak{m}^{i-s} N$ for all $i \geq s$.*
(iv) *$N \cap \mathfrak{m}^{s+1} F = \mathfrak{m} N$ and $\mu(N^*) = k$.*
(v) *The set $\{v_1, \ldots, v_k\}$ is a standard basis of N, and $\nu(v_i) = s$ for all $1 \leq i \leq k$.*

Proof.
(i)⇒(iii): This implication is the content of Lemma 3.3.
(iii)⇒(ii): We have

$$(G_{\mathcal{F}}(N))_i = (N \cap \mathfrak{m}^i F)/(N \cap \mathfrak{m}^{i+1} F) \text{ and } (G_{\mathfrak{m}}(N)(-s))_i = \mathfrak{m}^{i-s} N/\mathfrak{m}^{i-s+i} N.$$

Since $N \subset \mathfrak{m}^s F \setminus \mathfrak{m}^{s+1}F$, $(G_{\mathcal{F}}(N))_i = 0$ for $i < s$. Also, since $G_{\mathfrak{m}}(N)$ is generated in degree zero, $(G_{\mathfrak{m}}(N)(-s))_i = 0$ for $i < s$. Now, by (iii), for every $i \geq s$, the degree i components of $G_{\mathcal{F}}(N)$ and $G_{\mathfrak{m}}(N)(-s)$ are equal, which proves (ii).
(ii)$\Rightarrow$(i): This implication follows, since $G_{\mathfrak{m}}(N)$ is generated in degree 0.
(iii) $\Rightarrow$ (iv): Clearly, $N \cap \mathfrak{m}^{s+1}F = \mathfrak{m}N$. Now, note that $\mu(G_{\mathfrak{m}}(N)(-s)) = k$. Hence, by the implication (iii) $\Rightarrow$ (ii), we get $\mu(N^*) = k$.
(iv) $\Rightarrow$ (v): This implication is the content of Lemma 3.5.
(v) $\Rightarrow$ (i): Since $\{v_1^*, \dots, v_k^*\}$ is a generating set of $G_{\mathcal{F}}(N)$, with $\deg(v_j^*) = s$ for all $1 \leq j \leq k$, $G_{\mathcal{F}}(N)$ is equigenerated. □

Example 2.13 shows that even if R is Cohen-Macaulay and all entries in the presentation matrix of M have the same order, the associated graded module $G_{\mathfrak{m}}(M)$ need not have a pure first syzygy module. This shows that in statement (v) of the above theorem, the condition $\{v_1, \dots, v_k\}$ is a standard basis is necessary.

4. Free Resolutions over Associated Graded Rings

Lemma 4.1. *Let the notation be as in Remark 2.10, with ϕ_0 mapping minimally onto M. Suppose $N = \ker(\phi_0)$, and F_1 is a free R-module such that $\phi_1 : F_1 \to F_0$ maps minimally onto N. Then*

a) *ϵ is surjective, and $\Omega_1^A(G_{\mathfrak{m}}(M)) \simeq \ker(\epsilon)$. Moreover, $\epsilon \circ \phi_1^* = 0$.*
b) *If $\Omega_1^A(G_{\mathfrak{m}}(M))$ is equigenerated in degree s, then*
   i) *Every column of ϕ_1 has order s.*
   ii) *$\phi_1^* : G_{\mathfrak{m}}(F_1)(-s) \to G_{\mathfrak{m}}(F_0)$ maps minimally onto $\ker(\epsilon)$.*
   iii) *$\operatorname{Im}(\phi_1^*) = \ker(\epsilon) \simeq G_{\mathfrak{m}}(N)(-s)$.*

   In particular, $G_{\mathfrak{m}}(F_1)(-s) \xrightarrow{\phi_1^*} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \to 0$ is exact.

Proof. a) The map ϵ is surjective as $\{u_1^*, u_2^*, \dots, u_l^*\}$ is a generating set of $G_{\mathfrak{m}}(M)$. We also see that $\Omega_1^A(G_{\mathfrak{m}}(M)) \simeq \ker(\epsilon)$ since it is a minimal generating set.
Let $\phi_1 = (a_{ij})$ and $\nu(\phi_1) = s$. Then $\phi_1^* = (a_{ij} + \mathfrak{m}^{s+1})$. Since $\phi_0 \circ \phi_1 = 0$, we have $\sum_{i=1}^l a_{ij} u_i = 0$ for all j. Therefore,

$$\sum_{i=1}^{l} (a_{ij} + \mathfrak{m}^{s+1}) u_i^* = \sum_{i=1}^{l} (a_{ij} + \mathfrak{m}^{s+1})(u_i + \mathfrak{m}M) = \sum_{i=1}^{l} (a_{ij} u_i + \mathfrak{m}^{s+1} M) = 0$$

for all j. This proves that $\epsilon \circ \phi_1^* = 0$.
b) We know that N is generated minimally by the columns of ϕ_1, say $v_1, \dots, v_l$. By Lemma 2.11, we have $N^* = \ker(\epsilon) \simeq \Omega_1^A(G_{\mathfrak{m}}(M))$. By Lemma 3.1, $G_{\mathcal{F}}(N) \simeq N^*$. Since $\Omega_1^A(G_{\mathfrak{m}}(M))$ is equigenerated in degree s, by (i) $\Rightarrow$ (v) of Proposition 3.6, we get that all v_i have the same order s, and $\operatorname{Im}(\phi_i^*) = \{v_1^*, \dots, v_l^*\} = N^*$. The isomorphism $\ker(\epsilon) \simeq G_{\mathfrak{m}}(N)(-s)$ follows from (i) $\Rightarrow$ (ii) of Proposition 3.6. Finally, since $\mu(N) = \mu(G_{\mathfrak{m}}(N)(-s))$, we get that ϕ_1^* maps minimally onto $\ker(\epsilon)$, which completes the proof. □

Remark 4.2. Let

$$\mathbb{F}_\bullet : \ \cdots \to F_p \xrightarrow{\phi_p} \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

be a free resolution of an R-module M, where $\nu(\phi_i) = s_i$ for $i \geq 1$. Then we have a natural associated graded complex defined as follows:

$$\mathbb{F}_\bullet^* : \ \cdots \to G_{\mathfrak{m}}(F_p)(-\delta_p) \xrightarrow{\phi_p^*} \cdots \to G_{\mathfrak{m}}(F_1)(-\delta_1) \xrightarrow{\phi_1^*} G_{\mathfrak{m}}(F_0) \to 0,$$

where, $\delta_i = \sum_{j=1}^i s_j$.

Question 4.3. *Let $\widetilde{M} = \operatorname{coker}(\phi_1^*)$.*

a) *Is $\mathbb{F}_\bullet^*$ acyclic?*
b) *Is $\widetilde{M} \simeq G_{\mathfrak{m}}(M)$?*

Remark 4.4. a) If $\mathbb{F}_\bullet^*$ is acyclic, then $\widetilde{M}$ has a pure resolution of type $(\delta_0 = 0, \delta_1, \delta_2, \ldots)$ with $\beta^A_{i,\delta_i} = \beta^R_i(M)$.

b) With ϵ as in Remark 2.10, from Lemma 4.1, we have $\epsilon \circ \phi_1 = 0$. Therefore, $\widetilde{M}$ maps onto $G_\mathfrak{m}(M)$, and we have a short exact sequence $0 \to K \to \widetilde{M} \to G_\mathfrak{m}(M) \to 0$. Note that if $K = 0$, then Question 4.3 (b) has a positive answer.

In the next theorem we see a sufficient condition for exactness of $\mathbb{F}_\bullet^*$.

Theorem 4.5. *Let M be a finitely generated R-module such that $G_\mathfrak{m}(M)$ has a pure resolution over A. Then with notation as in the previous remark, $\mathbb{F}_\bullet^*$ is a minimal free resolution of $G_\mathfrak{m}(M)$.*

Proof. Denote ϵ as ϕ_0^*, let $\delta_0 = 0$, $K_0 = G_\mathfrak{m}(M)$, and $N_i = \operatorname{Im}(\phi_i)$, $K_{i+1} = \ker(\phi_i^*)$, for all $i \geq 0$. By induction on i, for all $i \geq 1$ we prove the following
Claim:
- (i) $0 \to K_i \to G_\mathfrak{m}(F_{i-1})(-\delta_{i-1}) \xrightarrow{\phi^*_{i-1}} K_{i-1} \to 0$ is exact,
- (ii) $K_i \simeq G_\mathfrak{m}(N_i)(-\delta_i) \simeq \Omega_i(G_\mathfrak{m}(M))$,
- (iii) ϕ_i^* maps minimally onto K_i.

Proof of claim. Since $\Omega_1^A(G_\mathfrak{m}(M))$ is equigenerated, the statements (i)-(iii) hold for $i = 1$ by Lemma 4.1 (b). Inductively assume that the statements (i)-(iii) hold for some $i \geq 1$.
Then ϕ_i^* maps minimally onto K_i. Since $K_{i+1} = \ker(\phi_i^*)$, we get that the sequence $0 \to K_{i+1} \to G_\mathfrak{m}(F_i)(-\delta_i) \xrightarrow{\phi_i^*} K_i \to 0$ is exact. Furthermore, the facts that $G_\mathfrak{m}(M)$ has a pure resolution and $K_i \simeq \Omega_i(G_\mathfrak{m}(M))$ imply that $K_{i+1} \simeq \Omega_{i+1}(G_\mathfrak{m}(M))$, and hence is equigenerated.
Now, consider the short exact sequence $0 \to N_{i+1} \to F_i \xrightarrow{\phi_i} N_i \to 0$. Since $K_i \simeq G_\mathfrak{m}(N_i)(-\delta_i)$, from the exact sequence above we have $K_{i+1} = \Omega_1(G_\mathfrak{m}(N_i))(-\delta_i)$. Since K_{i+1} is equigenerated, so is $\Omega_1(G_\mathfrak{m}(N_i)) \simeq K_{i+1}(\delta_i)$.
Hence, by Lemma 4.1, we get that $K_{i+1}(\delta_i) \simeq G_\mathfrak{m}(N_{i+1})(-s_{i+1})$, i.e., $K_{i+1} \simeq G_\mathfrak{m}(N_{i+1})(-\delta_{i+1})$. Moreover, $\phi^*_{i+1} : G_\mathfrak{m}(F_{i+1})(-s_{i+1}) \to G_\mathfrak{m}(F_i)$ maps minimally onto $K_{i+1}(\delta_i)$, or equivalently, $\phi^*_{i+1} : G_\mathfrak{m}(F_{i+1})(-\delta_{i+1}) \to G_\mathfrak{m}(F_i)(-\delta_i)$ maps minimally onto K_{i+1}. Hence, the claim is proved.
By the claim we have $\operatorname{Im}(\phi_i^*) = \ker(\phi^*_{i-1})$ for all $i \geq 1$, i.e., $\mathbb{F}_\bullet^*$ is exact. Moreover, since ϕ_i^* maps minimally onto K_i for all $i \geq 0$, we get that $\mathbb{F}_\bullet^*$ is a minimal free resolution of $G_\mathfrak{m}(M)$. ☐

Corollary 4.6. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring, and M be a finitely generated R-module. Consider a minimal free resolution*

$$\mathbb{F}_\bullet : \ \cdots \to F_p \xrightarrow{\phi_p} \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

of M, and let $\Omega_i = \Omega_i^R(M)$, $s_i = \nu(\phi_i)$, $\delta_0 = 0$, and $\delta_i = \sum_{j \leq i} s_j$ for all $i \geq 1$. Then the following are equivalent
- i) *$G_\mathfrak{m}(M)$ has a pure resolution.*
- ii) *For every $i \geq 1$ we have $\Omega_i \cap \mathfrak{m}^j F_{i-1} = \mathfrak{m}^{j-s_i}\Omega_i$ for all $j > s_i$.*
- iii) *For every $i \geq 1$ we have $\Omega_i \cap \mathfrak{m}^j F_{i-1} = \mathfrak{m}^{j-s_i}\Omega_i$ for all $s_i < j \leq \operatorname{reg}_A(G_\mathfrak{m}(M)) + i - \delta_{i-1}$.*

If this happens, then $G_\mathfrak{m}(M)$ is pure of type $\delta = (0, \delta_1, \delta_2, \ldots)$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $G_\mathfrak{m}(M)$ has a pure resolution. Then by Theorem 4.5, $\mathbb{F}_\bullet^*$ is a minimal free resolution of $G_\mathfrak{m}(M)$, and $\Omega_i^A(G_\mathfrak{m}(M)) \simeq G_\mathfrak{m}(\Omega_i)(-\delta_i)$, which is equigenerated. Now, by Lemma 3.3, $\Omega_i \cap \mathfrak{m}^j F_{i-1} = \mathfrak{m}^{j-s_i}\Omega_i$ for all $j \geq s_i$.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i): We induce on i to prove that $\Omega_i(G_\mathfrak{m}(M))$ is generated in degree δ_i. If $\Omega_1 \neq 0$, then note that by Lemma 3.4, the hypothesis implies that $\Omega_1^A(G_\mathfrak{m}(M))$ has no minimal generator in degree j for all $s_1 < j \leq \operatorname{reg}_A(G_\mathfrak{m}(M)) + 1$. By definition of regularity, every minimal generator of $\Omega_1(G_\mathfrak{m}(M))$ has degree at most $\operatorname{reg}_A(G_\mathfrak{m}(M)) + 1$. Hence, every minimal generator of $\Omega_1^A(G_\mathfrak{m}(M))$ has degree $s_1 = \delta_1$, which proves the result for $i = 1$.

Inductively assume that the result is true for some $i \geq 1$. Then $\Omega_i^A(G_{\mathfrak{m}}(M))$ is generated in degree δ_i. If $\Omega_{i+1} \neq 0$, then by Lemma 3.4, the hypothesis implies that $\Omega_{i+1}^A(G_{\mathfrak{m}}(M))$ has no minimal generator in degree j for all $\delta_{i+1} = s_{i+1} + \delta_i < j \leq \operatorname{reg}_A(G_{\mathfrak{m}}(M)) + (i+1)$. By definition of regularity, every minimal generator of $\Omega_{i+1}(G_{\mathfrak{m}}(M))$ has degree at most $\operatorname{reg}_A(G_{\mathfrak{m}}(M)) + (i+1)$. Hence, every minimal generator of $\Omega_{i+1}^A(G_{\mathfrak{m}}(M))$ has degree δ_{i+1} Therefore, by induction, it follows that $G_{\mathfrak{m}}(M)$ has pure resolution. □

Corollary 4.7. *If A is Koszul, then $\mathcal{P}_{\mathsf{k}}^R(z)$ is rational.*

Proof. If A is Koszul, then k has a linear A-resolution. Hence, by [8, Remark 7.4.4], we have $\mathcal{P}_{\mathsf{k}}^A(z) = 1/H_A(-z)$. Now, by Theorem 4.5 we have $\mathcal{P}_{\mathsf{k}}^R(z) = \mathcal{P}_{\mathsf{k}}^A(z)$. Hence $\mathcal{P}_{\mathsf{k}}^R(z) = 1/H_A(-z)$, proving the rationality of $\mathcal{P}_{\mathsf{k}}^R(z)$. □

Theorem 4.8. *Let M be an R-module with $\operatorname{pdim}_R(M) < \infty$ such that $G_{\mathfrak{m}}(M)$ has a pure A-resolution. Let*

$$\mathbb{F}_\bullet : 0 \to F_p \xrightarrow{\phi_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

be a minimal resolution of M with $\beta_i = \operatorname{rank}(F_i)$. Then

a) $\operatorname{codim}(M) \leq \operatorname{pdim}(M)$.
b) *If M is Cohen-Macaulay, then R is Cohen-Macaulay.*

Proof. Since $G_{\mathfrak{m}}(M)$ has a pure resolution, by Theorem 4.5, $\mathbb{F}_\bullet^*$ is a minimal free resolution of $G_{\mathfrak{m}}(M)$ of type $(0, \delta_1, \ldots, \delta_p, \infty, \infty, \ldots)$. In particular, $\operatorname{pdim}_A(G_{\mathfrak{m}}(M)) = \operatorname{pdim}_R(M) = p$. By [1, Proposition 3.7], we have $\operatorname{codim}(G_{\mathfrak{m}}(M)) \leq \operatorname{pdim}(G_{\mathfrak{m}}(M))$. Hence, $\operatorname{codim}(M) \leq \operatorname{pdim}(M)$. This proves (a).
Now, let M be Cohen-Macaulay. By the Auslander-Buchsbaum formula and (a), we have $\operatorname{depth}(R) = \operatorname{depth}(M) + \operatorname{pdim}_R(M) = \dim(M) + \operatorname{pdim}_R(M) \geq \dim(M) + \operatorname{codim}(M) = \dim(R)$. So, R is Cohen-Macaulay. □

To prove the next major result, we require the following lemma.

Lemma 4.9. *Let R be a Noetherian local ring and M be a Cohen-Macaulay R-module. If N is a nonzero submodule of M, then $\dim(N) = \dim(M)$.*

Proof. Note that since M is Cohen-Macaulay, $\dim(M) = \dim(R/\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Ass}(M)$. Also, $\operatorname{Ass}(N) \neq \emptyset$, since $N \neq 0$. Since $\operatorname{Ass}(N) \subset \operatorname{Ass}(M)$, and $\dim(N) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(N)\}$, we see that $\dim(N) = \dim(M)$. □

Definition 4.10. *Let R be a Noetherian local ring and M be a finitely generated R-module. Then the* Cohen-Macaulay defect *of M is defined as $\operatorname{cmd}(M) = \dim(M) - \operatorname{depth}(M)$.*

Note that $\operatorname{cmd}(M) = 0$ if and only if M is Cohen-Macaulay.

Theorem 4.11. *Let M be an R-module such that $\operatorname{pdim}_R(M) = p < \infty$ and $G_{\mathfrak{m}}(M)$ has a pure resolution. Let*

$$\mathbb{F}_\bullet : 0 \to F_p \xrightarrow{\phi_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

be a minimal resolution of M with $\beta_i = \operatorname{rank}(F_i)$. Then the following are equivalent:

i) $\operatorname{cmd}(M) = \operatorname{cmd}(R)$.
ii) $\beta_i = b_i\beta_0$ *for* $i = 1, \ldots, p$, *where* $b_i = (-1)^{i-1} \prod_{j \neq i} \frac{\delta_j}{\delta_j - \delta_i}$.
iii) $\operatorname{cmd}(G_{\mathfrak{m}}(M)) = \operatorname{cmd}(A)$.

Furthermore, if any of the above equivalent statements hold, then $e(M) = e(R)\dfrac{\beta_0}{p!} \prod_{i=1}^p \delta_i$.

Proof. Since $G_{\mathfrak{m}}(M)$ has a pure resolution, by Theorem 4.5, $\mathbb{F}_\bullet^*$ is a minimal pure resolution of $G_{\mathfrak{m}}(M)$ of type $(0, \delta_1, \ldots, \delta_p, \infty, \infty, \ldots)$ with $\beta_i^A(G_{\mathfrak{m}}(M)) = \beta_i$. Thus, from [1, Theorem 3.9], we get the equivalence of (ii) and (iii).

(i) $\Rightarrow$ (iii): Recall that $\dim(M) = \dim(G_{\mathfrak{m}}(M))$ (e.g., see [3, Theorem 4.5.6]). By the Auslander-Buchsbaum formula,

$$\begin{aligned}\dim(G_{\mathfrak{m}}(M)) - \operatorname{depth}(G_{\mathfrak{m}}(M)) &= \dim(M) - (\operatorname{depth}(A) - p)\\ &= \dim(M) - \operatorname{depth}(A) + \operatorname{depth}(R) - \operatorname{depth}(M)\\ &= \dim(R) - \operatorname{depth}(A)\\ &= \dim(A) - \operatorname{depth}(A),\end{aligned}$$

where the third and the fourth equalities follow since $\operatorname{cmd}(M) = \operatorname{cmd}(R)$, and $\dim(A) = \dim(R)$ respectively. Hence, $\operatorname{cmd}(G_{\mathfrak{m}}(M)) = \operatorname{cmd}(A)$.
(iii) $\Rightarrow$ (i): From $\operatorname{cmd}(G_{\mathfrak{m}}(M)) = \operatorname{cmd}(A)$ we have $\dim(M) = \dim(R) - \operatorname{depth}(A) + \operatorname{depth}(G_{\mathfrak{m}}(M))$. Also, since $\operatorname{pdim}_R(M) = \operatorname{pdim}(G_{\mathfrak{m}}(M))$, we have $\operatorname{depth}(M) = \operatorname{depth}(R) - \operatorname{depth}(A) + \operatorname{depth}(G_{\mathfrak{m}}(M))$. Thus, it follows that $\operatorname{cmd}(M) = \dim(M) - \operatorname{depth}(M) = \dim(R) - \operatorname{depth}(R) = \operatorname{cmd}(R)$.
Finally, if any of the conditions (i)-(iii) hold, then by [1, Corollary 4.1], we get $e(M) = e(R)\dfrac{\beta_0}{p!}\prod_{i=1}^p \delta_i$. □

Theorem 4.12. *Let M be an R-module with $\operatorname{pdim}(M) = p < \infty$. Let*

$$\mathbb{F}_\bullet : 0 \to F_p \xrightarrow{\phi_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

be a minimal resolution of M with $\beta_i = \operatorname{rank}(F_i)$. Then the following are equivalent:

i) *$G_{\mathfrak{m}}(M)$ has a pure resolution and is Cohen-Macaulay.*
ii) *A is Cohen-Macaulay and the following hold:*
   (a) *$\mathbb{F}_\bullet^*$ is acyclic.*
   (b) *$\beta_i = b_i\beta_0$ for $i = 1, \ldots, p$, where $b_i = (-1)^{i-1}\prod_{j\neq i}\frac{\delta_j}{\delta_j - \delta_i}$.*
   (c) *The multiplicity of M,*

$$e(M) = e(R)\frac{\beta_0}{p!}\prod_{i=1}^{p}\delta_i.$$

iii) *$G_{\mathfrak{m}}(M)$ has a pure resolution, and A and M are Cohen-Macaulay.*

Proof. (i) $\Rightarrow$ (ii): Since $G_{\mathfrak{m}}(M)$ has a pure resolution, by Theorem 4.5, $\mathbb{F}_\bullet^*$ is acyclic and it is a minimal pure resolution of $G_{\mathfrak{m}}(M)$ of type $(0, \delta_1, \ldots, \delta_p, \infty, \infty, \ldots)$ with $\beta_i^A(G_{\mathfrak{m}}(M)) = \beta_i$. Hence, by [1, Theorem 4.9], we get that A is Cohen-Macaulay. Since $G_{\mathfrak{m}}(M)$ is Cohen-Macaulay, the statements (b) and (c) hold by Theorem 4.11.
(ii) $\Rightarrow$ (iii): If $\mathbb{F}_\bullet^*$ is acyclic and the Betti numbers of M satisfy (b), then by Theorem 4.11 and the fact that A is Cohen-Macaulay, we get that $E = \operatorname{coker}(\phi_1^*)$ is Cohen-Macaulay of dimension $\dim(R) - p$ (by the Auslander-Buchsbaum formula). With $\epsilon : G_{\mathfrak{m}}(F_0) \to G_{\mathfrak{m}}(M)$ as in Remark 2.10, we have $\epsilon \circ \phi_1^* = 0$ by Lemma 4.1. Therefore, $\operatorname{Im}(\phi_1^*) \subset \ker(\epsilon)$, which gives us the short exact sequence

$$0 \to K \to E \to G_{\mathfrak{m}}(M) \to 0.$$

By [1, Corollary 4.1], we have $e(E) = e(R)\frac{\beta_0}{p!}\prod_{i=1}^p \delta_i$. We also have $e(G_{\mathfrak{m}}(M)) = e(M)$ by definition and hence, $e(G_{\mathfrak{m}}(M)) = e(R)\frac{\beta_0}{p!}\prod_{i=1}^p \delta_i = e(E)$. Note that $\dim(G_{\mathfrak{m}}(M)) = \dim(M) \geq \operatorname{depth}(M) = \operatorname{depth}(R) - p = \dim(R) - p = \dim(E)$. Since E maps onto $G_{\mathfrak{m}}(M)$, we have $\dim(E) = \dim(G_{\mathfrak{m}}(M))$. Note that $\dim(K) \leq \dim(E) = \dim(G_{\mathfrak{m}}(M))$. Then $e(G_{\mathfrak{m}}(M)) = e(E)$ forces $\dim(K) < \dim(E)$, for example, by using properties of the respective Hilbert series. Theorefore, $K = 0$ by Lemma 4.9. So, $\mathbb{F}_\bullet^*$ is a resolution of $G_{\mathfrak{m}}(M) \simeq E$, which is pure. Note that since A is Cohen-Macaulay, so is R. Since $G_{\mathfrak{m}}(M)$ has a pure resolution and A is Cohen-Macaulay, by (ii) $\Rightarrow$ (i) of Theorem 4.11, we get that M is Cohen-Macaulay.
(iii) $\Rightarrow$ (i): Since A is Cohen-Macaulay, so is R. Since $G_{\mathfrak{m}}(M)$ has a pure resolution and M is Cohen-Macaulay, by (i) $\Rightarrow$ (iii) of Theorem 4.11, we get that $G_{\mathfrak{m}}(M)$ is Cohen-Macaulay. □

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