# Free Resolutions and Associated Invariants

Master's Thesis

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# Summary

This report is divided into seven chapters. In chapter one, graded resolutions are introduced and some related fundamental results are proved. In chapter two, we discuss Gröbner bases and Schreyer's algorithm to compute a (not necessarily minimal) free resolution of a finitely generated module $M$ over a polynomial ring $S$ in finitely many variables. Hilbert's syzygy theorem follows as a corollary.

In the third chapter, we compare the homological invariants of an ideal with its initial ideal. We also introduce the concept of polarization, which, given a polynomial ideal, produces a related squarefree ideal in a larger polynomial ring, with the same homological invariants as the original ideal. We also introduce the lexsegment ideal $I^{\text{lex}}$ of a graded polynomial ideal $I$, and show that $S/I$ and $S/I^{\text{lex}}$ have the same Hilbert function. In chapter four, we prove the Auslander-Buchsbaum-Serre Theorem, which characterizes regular local rings. While doing so, we also prove that the Koszul complex is contained in the minimal resolution of $\mathsf{k}$ for any Noetherian local ring $(R, \mathfrak{m}, \mathsf{k})$.

In chapter five, we prove certain results on the existence of bounds on projective dimension and regularity of an ideal. We present a result by Burch which constructs ideals with arbitrarily large projective dimension, but generated by just 3 elements, in a Cohen-Macaualay ring. We also prove that Stillman's question on upper bounds on projective dimension is equivalent to a similar question on upper bounds on regularity. We end chapter five by discussing Koszul algebras and proving a result by Avramov and Eisenbud which states that the regularity of any module over a Koszul algebra is finite.

Chapter six discusses pure resolutions and begins with a theorem by Herzog and Kühl on when a Cohen-Macaulay module over a polynomial ring can have a pure resolution. We then discuss Herzog, Hibi and Zheng's results on when each power of a quadratic monomial ideal can have a linear resolution. We also discuss a couple of results by Puthenpurakal on when associated graded modules have pure resolutions. In chapter seven, we construct the Taylor's resolution, a (not necessarily minimal) free resolution of any monomial ideal. We also construct the Eliahou-Kervaire resolution, a minimal free resolution of any stable monomial ideal.

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# Chapter 1

# Graded Rings, Modules and Resolutions

## 1.1 Graded Rings and Modules

**Definition 1.1.1.** *Let $H$ be a cancellative monoid under addition. A ring $R$ is said to be $H$-**graded** if $R = \bigoplus_{i \in H} R_i$, where, each $R_i$ is an abelian group and $R_iR_j \subseteq R_{i+j}$, for all $i, j \in H$. For each $i$, $R_i$ is called the **homogeneous component of degree** $i$ of $R$ and the nonzero elements of $R_i$ are called **homogeneous elements of degree i**.*

**Remark 1.1.2.**
(i) For a cancellative monoid $H$, we denote its associated group by $G$.
(ii) By an ordered monoid we mean a cancellative monoid $H$ with an order $<$ satisfying: whenever $a < b$ in $H$, we have $a + c < b + c$ for all $c \in H$.
(iii) If $H$ is an ordered monoid, then we say that it is well ordered if every nonempty subset $S$ of $H$ which is bounded below has the least element in $S$.

**Definition 1.1.3.** *A module $M$ is called as a **graded module** over a graded ring $R$ if $M = \bigoplus_{i \in G} M_i$, as a direct sum of subgroups of $M$ and for all $i \in H, j \in G$, $R_iM_j \subseteq M_{i+j}$.*

**Definition 1.1.4.** *An ideal $J$ of a graded ring $R$ is said to be **graded** if it satifies any of the following equivalent conditions:*
*(i) If $f \in J$, then every homogeneous component of $f$ is in $J$.*
*(ii) $J = \oplus_{i \in N} J_i$, where $J_i = R_i \bigcap J$.*
*(iii) If $J'$ is the ideal generated by all homogeneous elements in $J$, then $J = J'$.*
*(iv) $J$ has a system of homogeneous generators.*

**Proposition 1.1.5.** *Given a graded ideal $I$ in a graded ring $R$, every associated prime of $I$ is also graded.*

*Proof.* Suppose $J = (I : x)$ is a prime ideal for some $x$ in $R$. Let $x = x_l + x_{l+1} + \cdots + x_k$ where $x_i \in R_i$, $l < k$ and $x_l, x_k$ are non-zero.

Let $y = y_t + y_{t+1} + \cdots + y_s \in J$, where $y_i \in R_i$, $t < s$ and $x_t, x_s$ are non-zero. If we show that $y_t \in J$, we are done by induction on $s - t$.
To see this, observe that we have $xy \in I$ and since $I$ is graded, the lowest graded component of $xy$, which is $x_l y_t$, belongs to $I$. Similarly, $x_{l+1}y_t + x_l y_{t+1} \in I$, and on multiplying by $y_t$, we get that $x_{l+1}y_t^2 \in I$. Continuing in this manner, we get that $x_{l+i}y_t^{i+1} \in I$ for all $i = 0, 1, \ldots, k-l$, which implies that $y_t^{k-l+1}x \in I$ and hence, $y_t^{k-l+1} \in J$. Since $J$ is prime, $y_t \in J$. Hence, $J$ is a graded ideal. □

**Definition 1.1.6.** *Let $R$ be a $H$-graded ring and $M = \bigoplus_{i \in G} M_i$ be a finitely generated $R$-module. Then we define an $R$-module $M(d)$ by $M(d) = \bigoplus_{i \in G} M_{i+d}$. $M(d)$ is called a **shifted $R$-module**.*

**Definition 1.1.7.** *Let $M = \bigoplus_{i \in G} M_i$, $M' = \bigoplus_{i \in G} M'_n$ be graded modules over $R$. An $R$-linear map $f : M \to M'$ is said to be a **graded map of degree** $d$ if $f(M_i) \subseteq M'_{i+d}$ for all $i \in G$. If $f$ has degree zero, we simply say that $f$ is a graded $R$-module homomorphism.*

**Proposition 1.1.8.** *Let $R$ be nonnegatively graded, $M, N$ be graded $R$-modules and $\phi : M \to N$ be a graded homomorphism of degree $d$. Then*
*(i)* $\ker(\phi)$ *is a graded submodule of $M$.*
*(ii)* $\operatorname{Im}(\phi)$ *is a graded submodule of $N$.*

*Proof.* (i) It is clear that $\ker(\phi)$ is a submodule of $M$ considered without grading. To show that $\ker(\phi)$ is graded, it suffices to show that if $x = x_r + \cdots + x_s$, is in $\ker(\phi)$, then each $x_i$ is in $\ker(\phi)$. We show that $x_r \in \ker(\phi)$ and by induction we will get that $x_i \in \ker(\phi)$ for all $i$. Note that $\phi(x_i) \in N_{i+d}$. Therefore $\phi(x_r) \in N_{r+d} \cap (N_{(r+1)+d} \oplus \cdots \oplus N_{s+d}) = 0$. This shows that $\phi(x_r) = 0$ as desired.
(ii) It is clear that $\operatorname{Im}(\phi)$ is a submodule of $N$ considered without grading. To show that $\operatorname{Im}(\phi)$ is graded, it suffices to show that if $y = y_r + \cdots + y_s$, is in $\operatorname{Im}(\phi)$, then each $y_i$ is in $\operatorname{Im}(\phi)$. Since $\phi(M_i) \subseteq N_{i+d}$ and $y \in \operatorname{Im}(\phi)$, there exists $x = x_{r-d} + \cdots + x_{s-d} \in M$ such that $\phi(x) = y$ and $\phi(x_{i-d}) = y_i$. This shows that $y_i \in \operatorname{Im}(\phi)$. This completes the proof. □

**Remark 1.1.9.**
(i) If $I$ is a graded ideal of $R$, then we have $R_i I_j \subseteq I_{i+j}$.
(ii) If $I$ is a graded ideal of $R$, then the quotient ring $R/I$ inherits the grading from $R$ by $(R/I)_i = R_i/I_i$.
(iii) If $N$ is a graded submodule of a graded module $M$, then $M/N$ is graded with the grading given by $(M/N)_i = M_i/N_i$.

**Proposition 1.1.10.** *Tensor products of graded $R$-modules is graded, i.e., if $M$ and $N$ are graded $R$-modules, then $M \otimes N$ is graded $R$-module.*

*Proof.* We know that $M \otimes N$ is an $R$-module. We give grading to $M \otimes N$ as follows:
Define $(M \otimes N)_i$ to be generated (as a $\mathbb{Z}$-module) by all the elements in $M \otimes N$ of the form $m \otimes n$,

where $\deg(m)+\deg(n)=i$. Then we have $M\otimes N=\bigoplus_{i\in G}(M\otimes N)_i$. Moreover, for any $r_i\in R_i$ and $m\oplus n\in(M\otimes N)_j$, we have $r(m\otimes n)=(rm)\otimes n$. Therefore

$$\deg(r(m\otimes n))=(i+\deg(m))+\deg(n)=i+j.$$

This shows that $R_i(M\otimes N)_j\subseteq(M\otimes N)_{i+j}$. Hence $M\otimes N$ is graded.

□

Let $\mathrm{Hom}_i(M,N)=\{\phi:M\to N\mid deg(\phi)=i\}$. Then we define ${}^*\mathrm{Hom}(M,N)=\bigoplus_{i\in G}\mathrm{Hom}_i(M,N)$.

**Remark 1.1.11.** In general, ${}^*\mathrm{Hom}(M,N)\neq\mathrm{Hom}(M,N)$. However, we have the equality in a special case which we will prove shortly.

**Lemma 1.1.12.** *Let $M=\bigoplus_{i=1}^{m}R(n_i)$ and $N$ be graded $R$-modules. Then ${}^*\mathrm{Hom}(M,N)\cong\mathrm{Hom}(M,N)$ with grading forgotten.*

*Proof.* It is clear that every $\phi=\phi_r+\cdots+\phi_s\in{}^*\mathrm{Hom}(M,N)$ is in $\mathrm{Hom}(M,N)$, and hence ${}^*\mathrm{Hom}(M,N)\subseteq\mathrm{Hom}(M,N)$. To show the other inclusion assume that $\phi\in\mathrm{Hom}(M,N)$. Let $e_j=(0,\ldots,0,1,0,\ldots,0)$ where 1 occurs at $j$th place. Then $M$ is a free $R$-module with basis $\{e_1,\ldots,e_m\}$. If $\phi(e_j)=y_{j1}+\cdots+y_{jr_j}\in N$, then we have

$$\phi=\phi_{11}+\cdots\phi_{1r_1}+\cdots+\phi_{m1}+\cdots+\phi_{mr_m}$$

where $\phi_{js}:M\to N$ is given by $\phi_{js}(e_j)=y_{js}$ and $\phi_{js}(e_i)=0$ for all $i\neq j$. Note that each $\phi_{js}$ is well defined since $\{e_1,\ldots,e_m\}$ is a basis for $M$. Moreover $\phi_{js}$ is a graded homomorphism of degree $\deg(y_{js})+n_j$. Therefore $\phi\in{}^*\mathrm{Hom}(M,N)$. This completes the proof. □

**Proposition 1.1.13.** *Let $R$ be a graded Noetherian ring, $M$ be a finitely generated graded $R$-module and $N$ be any graded $R$-module. Then ${}^*\mathrm{Hom}(M,N)=\mathrm{Hom}(M,N)$ with grading forgotten.*

*Proof.* It is clear that every $\phi=\phi_r+\cdots+\phi_s\in{}^*\mathrm{Hom}(M,N)$ is in $\mathrm{Hom}(M,N)$, and hence we have an inclusion ${}^*\mathrm{Hom}(M,N)\xrightarrow{i}\mathrm{Hom}(M,N)$.
Since $M$ is finitely generated and $R$ is Noetherian, we get an exact sequence of graded modules $G\to F\to M\to 0$ for some $F=\bigoplus_{j=1}^{n}R(n_j)$ and $G=\bigoplus_{j=1}^{m}R(m_j)$. By the previous lemma we have ${}^*\mathrm{Hom}(F,N)=\mathrm{Hom}(F,N)$, ${}^*\mathrm{Hom}(G,N)=\mathrm{Hom}(G,N)$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}0 & \longrightarrow & {}^*\mathrm{Hom}(M,N) & \longrightarrow & {}^*\mathrm{Hom}(F,N) & \longrightarrow & {}^*\mathrm{Hom}(G,N)\\ & & \downarrow{\scriptstyle i} & & \| & & \| \\ 0 & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & \mathrm{Hom}(F,N) & \longrightarrow & \mathrm{Hom}(G,N)\end{array}$$

Thus by five lemma, we get that the inclusion $i$ is an isomorphism. □

**Lemma 1.1.14** (Graded Nakayama Lemma)**.** *Let $H$ be an ordered monoid such that $i > 0$ for all $i \in H \setminus \{0\}$ and $R = \bigoplus_{i \in H} R_i$ be a graded ring. Let $M = \bigoplus_{i \in G} M_i$ be an $R$-module such that there exists $n \in G$ with $M_i = 0$ for all $i < n$. Further assume that $G$ is well ordered. If $R_+ = \bigoplus_{i \in H \setminus \{0\}} R_i$ and $R_+ M = M$ then $M = 0$.*

*Proof.* Let, if possible, $M \neq 0$. Let $m$ be the smallest element of $G$ such that for all $i < m$, we have $M_i = 0$ and $M_m \neq 0$. But then, $M = R_+ M \subseteq \bigoplus_{i\, m} M_i$, which has $m^{th}$ component equal to 0. This contradiction shows that $M = 0$. □

**Corollary 1.1.15.** *Let $R$ be a non negatively graded ring and $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. If $R_+ M = M$ then $M = 0$.*

*Proof.* Let $\{m_1, \ldots, m_r\}$ be a generating set for $M$ and $d$=min$\{\deg(m_i) \mid 1 \leq i \leq r\}$. Since $R$ is graded by $\mathbb{N} \cup \{0\}$, we get that $M_n = 0$, for every $n < d$. Thus, applying graded Nakayama lemma proved above, we get $M = 0$. □

## 1.2 Graded Resolutions

From now on we assume that $R$ is a graded ring with $R_0 = \mathsf{k}$, a field. We will mostly consider $R = \mathsf{k}[x_1, \ldots, x_r]$.

**Definition 1.2.1.** *Let $M$ be a graded $R$-module and*

$$F_\bullet : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

*be a free resolution of $M$. If all $F_i$'s are graded $R$-modules and all $\phi_i$'s are graded maps of degree zero, then we say that $F_\bullet$ is a **graded free resolution** of $M$.*

**Definition 1.2.2.** Let $R = \mathsf{k}[x_1, \ldots, x_n]$ and $M$ be a graded $R$-module. A graded free resolution

$$F_\bullet : \quad \cdots \to F_n \xrightarrow{\phi_n} \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

is said to be *minimal* if $\phi_i(F_i) \subseteq \langle x_1, \ldots, x_r \rangle F_{i-1}$ for all $i \geq 1$.

**Example 1.2.3.** Let $I = \langle x^2, y^2 \rangle$ and $R = \mathsf{k}[x, y]$. Then

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-2) \oplus R(-2) \xleftarrow{\phi_2} R(-4) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1, 0) = x^2$, $\phi_1(0, 1) = y^2$, $\phi_2(1) = (-y^2, x^2)$ is a minimal graded free resolution of $R/I$ over $R$.

**Example 1.2.4.** Let $I = \langle x^3, y^2 \rangle$ and $R = \mathsf{k}[x, y]$. Then

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\phi_0} R \xleftarrow{\phi_1} R(-3) \oplus R(-2) \xleftarrow{\phi_2} R(-5) \leftarrow 0,$$

where $\phi_0(1) = 1$, $\phi_1(1, 0) = x^3$, $\phi_1(0, 1) = y^2$, $\phi_2(1) = (-y^2, x^3)$ is a minimal graded free resolution of $R/I$ over $R$.

**Definition 1.2.5.** *Let $R = \mathsf{k}[x_1, \dots, x_n]$ and $M$ be a graded $R$-module.*

$$F_\bullet : \cdots \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

*be a minimal graded free resolution of $M$, where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}(M)}$. Then the numbers $\beta_{i,j}(M)$ are called **graded Betti numbers** of $M$. $\beta_i(M) = \sum_j \beta_{i,j}(M)$ is called the total ith Betti number of $M$.*

**Definition 1.2.6.** *Let $\beta_{i,j}$ be graded Betti numbers of $M$. Then **Betti table** of $M$ is written as*

| $j$ \ $i$ | 0 | 1 | $\cdots$ | $p$ | $\cdots$ |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\vdots$ |
| 0 | $\beta_{0,0}$ | $\beta_{1,1}$ | $\cdots$ | $\beta_{p,p}$ | $\vdots$ |
| 1 | $\beta_{0,1}$ | $\beta_{1,2}$ | $\cdots$ | $\beta_{p,p+1}$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\vdots$ |

**Definition 1.2.7.** *Let $\mathsf{k}$ be a field and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded module over the polynomial ring $\mathsf{k}[x_1, \dots, x_r]$. Then the function $H_M : \mathbb{Z} \to \mathbb{Z}$, given by $H_M(j) = \dim_k(M_j)$ is called as **Hilbert function of M**.*

Let $\mathsf{k}$ be a field and $R = \mathsf{k}[x_1, \dots, x_n]$. Let $a \in R \setminus \{0\}$ be such that $\deg(a) = d$. Since $a$ is a nonzerodivisor on $R$, we get an exact sequence of $R$-modules

$$0 \to R(-d) \xrightarrow{\cdot a} R \to R/\langle a \rangle \to 0.$$

Since $R$ is graded, for each $i$, we have an exact sequence of $k$-vector spaces

$$0 \to R(-d)_i \xrightarrow{\cdot a} R_i \to [R/\langle a \rangle]_i \to 0.$$

Now, using rank-nullity theorem for vector spaces, we get

$$\dim_k(R_i) = \dim_k((R(-d))_i) + \dim_k((R/\langle a \rangle)_i),$$

i.e.,

$$H_R(i) = H_{R(-d)}(i) + H_{R/\langle a \rangle}(i).$$

Therefore, $H_R(i) = H_R(i - d) + H_{R/\langle a \rangle}(i)$ or $H_{R/\langle a \rangle}(i) = H_R(i) - H_R(i - d)$.

**Definition 1.2.8.** *Given* $k, M$ *as above, define the* ***Hilbert series*** *of* $M$ *as* $H_M(t) = \sum_{j\geq 0} H_M(j)t^j$.

The next corollary follows from the above definition.

**Corollary 1.2.9.** $H_{R/\langle a\rangle}(t) = H_R(t)/(1-t)^d$.

**Example 1.2.10.** Let $R = \mathsf{k}[x, y]$ and $a = x^2$. In this case, for all $i \geq 0$, we have $H_R(i) = i + 1$. This is because the $i$th graded component of $R$, as a $\mathsf{k}$-vector space is has a basis $\{x^r y^{i-r} \mid 0 \leq r \leq i\}$. For the element $x^2$, we have $\deg(x^2) = 2$. Hence, by the formula above, we must have $H_{R/\langle x^2\rangle}(i) = (i+1) - (i-1) = 2$; which is true as $\{\overline{xy^{i-1}}, \overline{y^i}\}$ form a $\mathsf{k}$-vector space basis of $(R/\langle x^2\rangle)_i$.

**Proposition 1.2.11.** *Let* $M, N$ *be graded* $R$*-modules. Then* $\operatorname{Tor}_i^R(M, N)$ *is graded for all* $i$.

*Proof.* Consider a graded free resolution of $M$ as follows:

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

Tensoring with $N$ gives a complex of graded modules

$$\cdots \to F_2 \otimes N \to F_1 \otimes N \to F_0 \otimes N \to M \otimes N \to 0.$$

Since $\operatorname{Tor}_i^R(M, N)$ is quotient of a graded submodule of a graded module by a graded submodule, we conclude that $\operatorname{Tor}_i^R(M, N)$ is graded for all $i$. $\square$

**Remark 1.2.12.** If $F_\bullet$ is a graded free resolution of $M$ then we define ${}^*\operatorname{Ext}_R^i(M, N) \cong H^i({}^*\operatorname{Hom}_R(F_\bullet, N))$. Then, by Proposition 1.1.13, if $R$ is Noetherian local ring and $M$ is finitely generated $R$-module, then ${}^*\operatorname{Ext}_R^i(M, N) \cong \operatorname{Ext}_R^i(M, N)$.

# Chapter 2

# Gröbner Bases and Schreyer's Algorithm

Let $\mathsf{k}$ be a field and $S = \mathsf{k}[x_1, \ldots, x_r]$.
If $a = (a_1, \ldots, a_r)$, $x^a$ will denote the monomial $x_1^{a_1} \ldots x_r^{a_r}$. As is convention, an ideal of $S$ generated by monomials will be referred to as a monomial ideal.

**Definition 2.0.1.** *Let $F$ be a finitely generated free module over $S$ with basis $\{e_1, \ldots, e_n\}$.*
*A* ***monomial*** *in $F$ is an element of the form $m = x^a e_i$ for some $i$. We say that such an $m$ involves the basis element $e_i$.*
*A* ***monomial submodule*** *of $F$ is a submodule generated by elements of this form. Any monomial submodule $M$ of $F$ may be written as*

$$M = \oplus I_j e_j \subseteq \oplus S e_j = F,$$

*with $I_j$ the monomial ideal generated by those monomials $m$ such that $me_j \in M$.*
*A* ***term*** *in $F$ is a monomial multiplied by a scalar.*

**Definition 2.0.2.** *Let $F$ be a finitely generated free module over $S$ with basis $\{e_1, \ldots, e_n\}$.*
*If $m$, $n$ are monomials of $S$, $u, v \in k$, and $v \neq 0$, then we say that the term $ume_i$ is divisible by the term $vne_j$ if $i = j$ and $m$ is* ***divisible*** *by $n$ in $S$; the quotient is then $um/vn \in S$.*

**Definition 2.0.3.** *The set of monomials in $M$ that are minimal elements in the partial order by divisibility on the monomials of $F$ are referred as* ***minimal generators of*** *$M$.*

## 2.1 Hilbert Function of Monomial Submodules

Let $F$ be a free $S$-module with basis $\{e_i : i = 1, \ldots, n\}$, and let $M \subseteq F$ be a monomial submodule. Since, as seen before, $M = \oplus I_j e_j$, we have $F/M = \oplus S/I_j$ and, since the Hilbert function is additive, it suffices to handle the case $F = S$ and $M = I$, where $I$ is a monomial ideal.
Choosing one of the monomial generators $f$ of $I$, and letting $I'$ be the monomial ideal generated by the remaining generators, we have the following graded exact sequence:

$$0 \to S/(I' : f)(-d) \xrightarrow{\cdot f} S/I' \to S/I \to 0,$$

where $d$ is the degree of $f$. If $I' = (f_1, f_2 \ldots, f_t)$, then

$$(I' : f) = (f_1/GCD(f_1, f), f_2/GCD(f_2, f), \ldots, f_n/GCD(f_t, f)).$$

For every integer $n$,

$$H_{S/I}(n) = H_{S/I'}(n) - H_{S/(I':f)}(n).$$

Note that both $I'$ and $(I' : f)$ have fewer minimal generators than $I$, and hence, using induction, we can compute an expression for the Hilbert function or polynomial of $I$.
By choosing $f$ sensibly, we can make the process much faster: If $f$ contains the largest power of some variable $x_1$ of any of the minimal generators of I, then the minimal generators of the resulting ideal $(I' : f)$ will not involve $x_1$ at all. They will thus involve strictly fewer of the variables than the number involved in the minimal generators of $I$.

## 2.2 Syzygies of Monomial Submodules

Let $F$ be a free module and let $M$ be a submodule of $F$ generated by monomials $m_1, \ldots, m_t$. Define

$$\phi : \oplus_{j=1}^{t} S\epsilon_j \to F; \phi(\epsilon_j) = m_j.$$

For each pair of indices $i, j$ such that $m_i$ and $m_j$ involve the same basis element of $F$, we define

$$m_{ij} = m_i/\mathrm{GCD}(m_i, m_j),$$

and we define $\sigma_{ij}$ to be the element of $\ker(\phi)$ given by

$$\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j.$$

**Lemma 2.2.1.** *With notation as above,* $\ker(\phi)$ *is generated by the set of all* $\sigma_{ij}$*, wherever defined.*

*Proof.* As a vector space over $k$, $\ker(\phi) = \oplus_f \ker(\phi)_f$, where

$$\ker(\phi)_f = \left\{ \sum_i a_i f_i \epsilon_i \in \ker(\phi) : m_i \textit{ divides } f,\ f_i = f/m_i,\ a_i \in \mathsf{k} \right\}.$$

Indeed, let

$$\sigma = \sum_i p_i \epsilon_i \in \ker(\phi).$$

For any monomial $f$ that occurs in one of the $p_j m_j$, and for each $i$, let $p_{i,f}$ be the term of $p_i$ such that $p_{i,f} m_i$ is a scalar times $f$. Then,

$$\sum_i p_i m_i = 0 \implies \sum_i \sum_f p_{i,f} m_i = 0 \implies \sum_f \sum_i p_{i,f} m_i = 0 \implies \forall f, \sum_i p_{i,f} m_i = 0.$$

Therefore, for all monomials $f$, $\sum_i p_{i,f}\epsilon_i \in \ker(\phi)$.
We may now assume $\sigma = \sum_i a_i f_i \epsilon_i$ for some monomial $f$ of $F$. If $\sigma = 0$, $\sigma$ lies in the module generated by $\sigma_{ij}$. If $\sigma \neq 0$, at least two of the $a_i f_i$ must be non-zero, since $\sum_i a_i f_i m_i = 0$. This implies that for some $i, j$, both $m_i$ and $m_j$ must divide $f$ and in fact, $m_i f_i = m_j f_j = f$, which implies that $m_{ji} = m_j/\text{GCD}(m_i, m_j)$ divides $f_i$. Let $k = f_i/m_{ji}$, then $k\sigma_{ij} \in ker(\phi)_f$, and $\sigma - a_i k\sigma_{ij}$ has fewer non-zero terms than $\sigma$. Hence, the proof is complete by induction on number of non-zero terms of $\sigma$. □

**Example 2.2.2.** *Let $S = \mathsf{k}[x, y]$, $F = S^2$, $M = \langle (x^2, 0), (0, xy), (0, y^3) \rangle$. Then we have*

$$\phi : \oplus_{j=1}^{3} S\epsilon_j \to F; \phi(\epsilon_1) = (x^2, 0), \phi(\epsilon_2) = (0, xy), \phi(\epsilon_3) = (0, y^3).$$

*Suppose for some $a_1, a_2, a_3 \in S$, $\phi(a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3) = 0$, then we have $(a_1x^2, a_2xy + a_3y^3) = 0$, and hence, $a_1 = 0$, $a_2 = by^2$, $a_3 = -bx$. Thus, $a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 = b(0, y^2 - x) = b\sigma_{23}$.*

## 2.3 Monomial Orders

Let $I$ be an ideal of $S$, $J$ be a monomial ideal of $S$ and $B$ be the set of all monomials not in $J$. Then, the elements of $B$ are $\mathsf{k}$-linearly independent modulo $I$ if and only if $J$ contains at least one monomial from every polynomial in $I$.
Indeed, suppose $J$ contains no monomial of $f \in I$, $f \neq 0$. Then $f \in \text{Span}(B) \cap I$, which implies that the elements of $B$ are linearly dependent modulo $I$. Conversely, suppose there exist $a_1, \ldots, a_n \in \mathsf{k}$ and $m_1, \ldots, m_n \in B$ such that $\sum_{i=1}^{n} a_i m_i \in I$, then $\sum_{i=1}^{n} a_i m_i$ is a polynomial in $I$ for which no monomials belong to $J$.
Moreover, if $B$ is a basis of $S/I$, $J$ must be a minimal monomial ideal containing at least one monomial from every polynomial in $I$. Indeed, suppose $J$ contains at least one monomial from each polynomial in $I$, but is not a minimal ideal satisfying this condition. Let $J_1 \subsetneq J$ satisfying the condition, and let $f \in J \setminus J_1$, where $f$ is a monomial. Suppose $f \in \text{Span}(B)$, that is, there exist $a_1, \ldots, a_n \in k$ and $m_1, \ldots, m_n \in B$ such that $f - \sum_{i=1}^{n} a_i m_i \in I$. Since $J_1$ contains at least one monomial of every polynomial in $I$, we have a contradiction. Hence, $B$ cannot span $S/I$ if $J$ is not the minimal monomial ideal containing one monomial from each polynomial in $I$.

**Definition 2.3.1.** *Let $F$ be a free $S$-module. A* ***monomial order*** *on $F$ is a total order $\tau$ on the monomials of $F$ such that the following two conditions are satisfied:*
*(i) if $m_1$ is a monomial of $F$ and $f \neq 1$ is a monomial of $S$, then $fm_1 >_\tau m_1$.*
*(ii) if $m_1$, $m_2$ are monomials of $F$ and $f \neq 1$ is a monomial of $S$, then $m_1 >_\tau m_2$ implies $fm_1 >_\tau fm_2$.*

**Lemma 2.3.2** (Well-Ordering Property)**.** *Let $F$ be a free $S$-module. The set of monomials in $F$ is well-ordered with respect to any monomial order, that is, every non-empty subset of monomials in $F$ has a least element.*

*Proof.* Let $X \subseteq F$ be a set of monomials. Since $S$ is Noetherian, the submodule of $F$ generated by $X$ must be generated by a finite subset of $X$, say, $Y$. Since $Y$ is a finite set of monomials, it must have a least element with respect to a monomial order. The least element of $Y$ must be the least element of $X$ because every element of $X$ is an element in $Y$ multiplied by a monomial in $S$. □

We will extend this notation to terms: If $um_1$ and $vm_2$ are terms with $0 \neq u, v \in \mathsf{k}$, and $m_1, m_2$ are monomials with $m_1 >_\tau m_2$ then we say $um_1 >_\tau vm_2$.

**Definition 2.3.3.** *Let $F$ be a free $S$-module and $\tau$ be a monomial order on $F$. For any $f \in F$, we define the **initial term** of $f$, denoted by $\text{in}_\tau(f)$ to be the greatest term of $f$ with respect to the order $\tau$. Given a submodule $M$ of $F$, define the **initial submodule** of $M$, denoted by $\text{in}_\tau(M)$, to be the monomial submodule generated by $\text{in}_\tau(f)$ for all $f \in M$.*

**Theorem 2.3.4** (Macaulay)**.** *Let $F$ be a free $S$-module and $M$ be a submodule of $F$. For any monomial order $\tau$ on $F$, the set $B$ of all monomials not in $\text{in}_\tau(M)$ forms a $\mathsf{k}$-basis for $F/M$.*

*Proof.* Suppose the set $B$ is not linearly independent. Then there exist distinct $m_1, \ldots, m_t \in B$ and $(a_1, \ldots, a_t) \in \mathsf{k}^\mathsf{t} \backslash \{0\}$ such that $f := a_1 m_1 + \cdots + a_t m_t \in M$. Since $\text{in}(f) \in \text{in}(M)$, there must exist $i \in \{1, \ldots, t\}$ such that $m_i \in \text{in}(M)$, which is a contradiction.
Suppose $B$ does not span $F/M$. Let $f \in F \backslash (M + \text{Span}(B))$ such that $f$ has minimal initial term among all elements of $F \backslash (M + \text{Span}(B))$. We can choose such an $f$ by the well-ordering property. If $\text{in}(f) \in \text{Span}(B)$, $f - \text{in}(f) \in F \backslash (M + \text{Span}(B))$ has smaller initial term than $f$. Hence, $\text{in}(f) \in \text{in}(M)$. However, this implies that there exists $g \in M$ such that $\text{in}(f) = \text{in}(g)$, and $f - g \in F \backslash (M + \text{Span}(B))$ has smaller initial term than $f$, leading to a contradiction. □

**Corollary 2.3.5.** *Given $F, M, \tau$ as above,* $\dim_\mathsf{k}(F/M) = \dim_\mathsf{k}(F/in_\tau(M))$.

**Corollary 2.3.6.** *Given monomial orders $\tau, \gamma$ on $S$ and an ideal $I \in S$ such that $\text{in}_\tau(I) \subset \text{in}_\gamma(I)$, we have $\text{in}_\tau(I) = \text{in}_\gamma(I)$.*

*Proof.* If $\text{in}_\tau(I) \subsetneq \text{in}_\gamma(I)$, the set of monomials in $S \backslash \text{in}_\gamma(I)$ is a proper subset of the set of monomials in $S \backslash \text{in}_\tau(I)$. However, both these sets of monomials form a $K$-basis of $S/I$, which is a contradiction. □

Here are some important examples of monomial orders when $F = S$. Let $a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r)$ and $m = x^a, m' = x^b$

**Lexicographic order:** $m >_{lex} m'$ if and only if $a_i > b_i$ for the smallest $i$ such that $a_i \neq b_i$.
**Graded lexicographic order:** $m >_{grlex} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i > b_i$ for the smallest $i$ such that $a_i \neq b_i$.
**Reverse graded lexicographic order:** $m >_{grevlex} m'$ if and only if $\deg(m) > \deg(n)$ or $\deg(m) = \deg(n)$ and $a_i < b_i$ for the largest $i$ such that $a_i \neq b_i$.

**Remark 2.3.7.** A "reverse lexicographic order" is not a monomial order, because 1 is not the least monomial. In fact, 1 is the largest monomial.

**Theorem 2.3.8.** *Every ideal $I \subset S = \mathsf{k}[x_1, \ldots, x_n]$ has only finitely many distinct initial ideals.*

*Proof.* Suppose $I$ has an infinite set $\Sigma_0$ of distinct initial ideals. Choose $f_1 \in I$, $f_1 \neq 0$. Since $f_1$ has only finitely many terms and since each element of $\Sigma_0$ contains at least one term of $f_1$, there exists a monomial $m_1$ in $f_1$ such that the set $\Sigma_1 := \{J \in \Sigma_0 : m_1 \in J\}$ is infinite. Hence, $\langle m_1 \rangle$ is strictly contained in an initial ideal of $I$, and by Theorem 2.3.4, the monomials outside $\langle m_1 \rangle$ are $\mathsf{k}$-linearly dependent modulo $I$. Thus, there exists a non-zero polynomial $f_2 \in I$ none of whose terms lies in $\langle m_1 \rangle$. Since $f_2$ has finitely many terms, there exists a monomial $m_2$ in $f_2$ such that the set $\Sigma_2 := \{J \in \Sigma_1 : m_1 \in J\}$ is infinite. Hence, $\langle m_1, m_2 \rangle$ is strictly contained in an initial ideal of $I$, and by Theorem 2.3.4, the monomials outside $\langle m_1, m_2 \rangle$ are $\mathsf{k}$-linearly dependent modulo $I$. Thus, there exists a non-zero polynomial $f_3 \in I$ none of whose terms lies in $\langle m_1, m_2 \rangle$. Now we can choose a monomial $m_3$ in $f_3$ such that the set $\Sigma_3 := \{J \in \Sigma_2 : m_1 \in J\}$ is infinite. Iterating this construction, we obtain an infinite strictly increasing chain of ideals in $S$:

$$\langle m_1 \rangle \subset \langle m_1, m_2 \rangle \subset \langle m_1, m_2, m_3 \rangle \subset \ldots$$

Since $S$ is Noetherian, we have a contradiction. ☐

**Definition 2.3.9.** *A finite subset $\mathcal{U} \in I$ is called a universal Gröbner basis if $\mathcal{U}$ is a Gröbner basis if $\mathcal{U}$ is a Gröbner basis of $I$ with respect to all monomial orders.*

**Theorem 2.3.10.** *Every ideal $S$ possesses a finite universal Gröbner basis $\mathcal{U}$.*

*Proof.* By Theorem 2.3.8, there are only finitely many distinct initial ideals of $I$. We can choose a reduced Gröbner basis for each initial ideal of $I$. Their union is finite, and is a universal Gröbner basis for $I$. ☐

**Definition 2.3.11.** *A **Gröbner basis** with respect to an order $\tau$ on a free module $F$ is a set of elements $g_1, \ldots, g_t \in F$ such that if $M$ is the submodule of $F$ generated by $g_1, ..., g_t$, then $\text{in}_\tau(g_1), ..., \text{in}_\tau(g_t)$ generate $\text{in}_\tau(M)$. We then say that $g_1, ..., g_t$ is a **Gröbner basis of** $M$.*

There is a Gröbner basis of any submodule $M$ of $F$, with respect to any monomial order: if $g_1, \ldots, g_t$ is a set of generators of $M$ which is not a Gröbner basis, we can adjoin $g_{t+1}, \ldots, g_{t'}$ until $\text{in}(g_1), \ldots, \text{in}(g_{t'})$ generate $\text{in}(M)$ (note that the Hilbert basis theorem implies that this can be done).

**Lemma 2.3.12.** *Let $N \subset M \subset F$ be submodules such that $\text{in}(N) = \text{in}(M)$ with respect to a given monomial order. Then, $N = M$.*

*Proof.* Suppose $N \neq M$, then, by the well-ordering property, there exists $f \in M \backslash N$ such that $f$ has the least initial term among all the elements of $M$ not in $N$. Since $f \in M$, we have $\text{in}(f) \in \text{in}(M) = \text{in}(N)$, which implies the existence of $g \in N$ such that $\text{in}(f) = \text{in}(g)$. Note that $f - g \in M \backslash N$, but has smaller initial term than $f$, which is a contradiction to the choice of $f$. ☐

The above lemma tells us that if $\langle \text{in}(g_1), \ldots, \text{in}(g_t) \rangle = \text{in}(M)$ for $g_1, \ldots, g_t \in M$, then $\langle g_1, \ldots, g_t \rangle = M$. This follows since $\langle \text{in}_(g_1), \ldots, \text{in}_(g_t) \rangle \subset \text{in}(\langle g_1, \ldots, g_t \rangle) \subset \text{in}(M)$.

## 2.4 Computing Syzygies

**Proposition 2.4.1** (Division Algorithm)**.** *Let $F$ be a free $S$-module with monomial order $\tau$. If $f, g_1, ..., g_t \in F$, then there is an expression*

$$f = \sum_{i=1}^{t} f_i g_i + f' \text{ with } f' \in F, f_i \in S,$$

*where none of the monomials of $f'$ is in $\langle \mathrm{in}(g_1), \ldots, \mathrm{in}(g_t) \rangle$ and $\mathrm{in}(f) \geq_\tau \mathrm{in}(f_i g_i)$ for every $i$.*

**Definition 2.4.2.** *With notation as above, any such $f'$ is called a **remainder** of $f$ with respect to $g_1, ..., g_t$, and an expression $f = \sum f_i g_i + f'$ satisfying the condition of the proposition is called a **standard expression** for $f$ in terms of the $g_i$.*

The proof outlines an algorithm to attain a standard expression for any $f \in F$.

*Proof.* If $f, g_1, \ldots, g_t \in F$, then we may produce a standard expression

$$f = \sum m_u g_{s_u} + f'$$

for $f$ with respect to $g_1, \ldots, g_t$ by defining the indices $s_u$ and the terms $m_u$ inductively. Having chosen $s_1, \ldots, s_p$ and $m_1, \ldots, m_p$, if

$$f'_p := f - \sum_{u=1}^{p} m_u g_{s_u} \neq 0$$

and $m$ is the maximal term of $f'_p$; that is divisible by $\mathrm{in}(g_i)$ for some $i$, then choose $s_{p+1} = i, m_{p+1} = m/\mathrm{in}(g_i)$. This process terminates when either $f'_p = 0$ or no $\mathrm{in}(g_i)$ divides a monomial of $f$; the remainder $f'$ is then the last $f'_p$ produced.
Note that the well-ordering property guarantees that this process must terminate, because the maximal term of $f'_p$ divisible by some $g_i$ decreases at each step. □

Fix the following notation:
Let $F$ be a free module over $S$ with monomial order $\tau$. Let $g_1, \ldots, g_t$ be non-zero elements of $F$, and let $\oplus S\epsilon_i$ be a free module with basis $\{\epsilon_1, \ldots, \epsilon_t\}$.
For two terms $m_1, m_2 \in F$, $m_1 < m_2$ denotes that the monomial corresponding to $m_1$ is less than the monomial corresponding to $m_2$ with respect to the order $\tau$.
For each pair of indices $i, j$ such that $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element of $F$, we define

$$m_{ij} = \mathrm{in}(g_i)/\mathrm{GCD}(\mathrm{in}(g_i), \mathrm{in}(g_j)) \in S,$$

and we set $\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j$ for $i < j$.

For each such pair $i, j$, choose a standard expression

$$m_{ji}g_i - m_{ij}g_j = \sum_{u=1}^{t} f_u^{(ij)} g_u + h_{ij}$$

for $m_{ji}g_i - m_{ij}g_j$ with respect to $g_1, \ldots, g_t$. Note that $\text{in}(f_u^{(ij)} g_u) < \text{in}(m_{ji}g_i)$.
Set $h_{ij} = 0$ if $\text{in}(g_i)$ and $\text{in}(g_j)$ involve different basis elements of $F$.
Define $\phi : \oplus S\epsilon_i \to F$, $\phi(\epsilon_i) = g_i$. Then, the set of $\sigma_{ij}$ generate the syzygies on the module generated by the elements $\text{in}(g_i)$ (by Lemma 2.2.1). Note that $\phi(\sigma_{ij}) = m_{ji}g_i - m_{ij}g_j$.

**Theorem 2.4.3** (Buchberger's Criterion)**.** *The elements $g_1, \ldots, g_t$ form a Gröbner basis if and only if $h_{ij} = 0$ for all $i$ and $j$.*

*Proof.* Let $M = \langle g_1, \ldots, g_t \rangle \subset F$. The expression for $h_{ij}$ implies that $h_{ij} \in M$, and hence $\text{in}(h_{ij}) \in \text{in}(M)$. However, if $g_1, \ldots, g_t$ is a Gröbner basis, the definition of a standard expression forces $h_{ij} = 0$ for all $i, j$.
Conversely, suppose that $h_{ij} = 0$ for all $i, j$. Let $f = \sum_{i=1}^{t} h_i g_i \in M$, where, among all possible $h_1, \ldots, h_t$ such that $f = \sum_{i=1}^{t} h_i g_i$, $h_1, \ldots, h_t$ are chosen such that $\max\{\text{in}(h_i g_i) : 1 \le i \le t\}$ is minimal. We prove that $\text{in}(f) \in \langle \text{in}(g_1), \ldots, \text{in}(g_t) \rangle$.
If $\text{in}(f) = \text{in}(h_i g_i)$ for some $i$, $\text{in}(g_i) | \text{in}(f) \Rightarrow \text{in}(f) \in \langle \text{in}(g_1), \ldots, \text{in}(g_t) \rangle$.
Hence, let $\text{in}(f) < \max\{\text{in}(h_i g_i) : 1 \le i \le t\} = m$. Define an equivalence relation $\equiv$ on terms as follows: $m_1 \equiv m_2$ if there exists $\lambda \in \mathsf{k} \backslash \{0\}$ such that $m_1 = \lambda m_2$. Without loss of generality, suppose $\text{in}(h_i g_i) \equiv m$ for $i = 1, \ldots, t_1$ and $\text{in}(h_i g_i) < m$ for $i = t_1 + 1, \ldots, t$

$$\begin{aligned} f &= \sum_{i=1}^{t} h_i g_i = \sum_{i=1}^{t_1} h_i g_i + \sum_{i=t_1}^{t} h_i g_i \\ &= \sum_{i=1}^{t_1} \text{in}(h_i) g_i + \sum_{i=1}^{t_1} (h_i - \text{in}(h_i)) g_i + \sum_{i=t_1+1}^{t} h_i g_i. \end{aligned}$$

Note that $\sum_{i=1}^{t_1} \text{in}(h_i)\text{in}(g_i) = 0$.
Define $\phi_1 : \oplus S\epsilon_j \to M, \phi_1(\epsilon_j) = \text{in}(g_j)$ and $\phi_2 : \oplus S\epsilon_j \to M, \phi_2(\epsilon_j) = g_j$. Note that $\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i \in \ker(\phi_1)$. Therefore, by Lemma 2.2.1,

$$\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i = \sum_{i<j} k_{ij}\sigma_{ij},$$ [1]

where $k_{ij} = a_{ij} m / \text{LCM}(\text{in}(g_i), \text{in}(g_j))$ for some $a_{ij} \in \mathsf{k}$.
Note that $\phi_2(\sum_{i=1}^{t_1} \text{in}(h_i)\epsilon_i) = \sum_{i=1}^{t_1} \text{in}(h_i) g_i$.

[1]let $k_{ij} = 0$ and $\sigma_{ij} = 0$ for $i, j$ where $\sigma_{ij}$ is not originally defined

Hence,

$$\sum_{i=1}^{t_1} \mathrm{in}(h_i)g_i = \sum_{i<j} k_{ij}(m_{ji}g_i - m_{ij}g_j) = \sum_{i<j} k_{ij} \sum_{u=1}^{t} f_u^{(ij)} g_u,$$

since $h_{ij} = 0$ for all $i, j$. Note that since $\mathrm{in}(f_u^{(ij)} g_u) < \mathrm{in}(m_{ji}g_i)$, we have $\mathrm{in}(k_{ij}f_u^{(ij)}) < m$.
Hence, we have an expression for $f = \sum_i h_i' g_i$, where $\max\{\mathrm{in}(h_i' g_i) : 1 \leq i \leq t\} < m$, which is a contradiction. □

This result gives us an effective method for computing Gröbner bases.

**Buchberger's Algorithm**: In the situation of Theorem 2.4.3, suppose that $M$ is a submodule of $F$, and let $g_1, \dots, g_t \in M$ be a set of generators of $M$. Compute the remainders $h_{ij}$. If all the $h_{ij} = 0$, then $\{g_1, \dots, g_t\}$ forms a Gröbner basis of $M$. If some $h_{ij} \neq 0$, then replace $g_1, \dots, g_t$ with $g_1, \dots, g_t, h_{ij}$, and repeat the process. As the submodule generated by the initial forms of $g_1, \dots, g_t, h_{ij}$ is strictly larger than that generated by the initial forms of $g_1, \dots, g_t$, this process must terminate after finitely many steps.

The next theorem shows that if $\{g_1, \dots, g_t\}$ is a Gröbner basis of $M$, the equations $h_{ij} = 0$ generate the first syzygy of $M$.
For $i < j$ such that $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element of $F$, we set

$$w_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j - \sum_{u=1}^{t} f_u^{(ij)} \epsilon_u.$$

Let $W$ be the set of all such $w_{ij}$.

**Theorem 2.4.4** (Schreyer). *With notation as above, suppose that $\{g_1, \dots, g_t\}$ is a Gröbner basis of $M$. Let $\gamma$ be the monomial order on $\oplus_{j=1}^{t} S\epsilon_j$ defined by taking $m\epsilon_u > n\epsilon_v$ if and only if*

$$\mathrm{in}(mg_u) >_\tau \mathrm{in}(ng_v) \textit{ with respect to the given order } \tau \textit{ on } F$$

*or*

$$\mathrm{in}(mg_u) \equiv \mathrm{in}(ng_v), \textit{ but } u < v.$$

*$W$ generates the first syzygy of $M$. Moreover, $W$ forms a Gröbner basis of the syzygies with respect to the order $\gamma$, and $\mathrm{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.*

*Proof.* We first prove that $\mathrm{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$. Since

$$\mathrm{in}(m_{ji}g_i) = \mathrm{in}(m_{ij}g_j),$$

and these terms are by hypothesis greater than any that appear in the $\sum_{u=1}^{t} f_u^{(ij)} g_u$, $\mathrm{in}(w_{ij})$ must be either $m_{ji}\epsilon_i$ or $-m_{ij}\epsilon_j$. Since $i < j$, $\mathrm{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.

To show that $W$ forms a Gröbner basis, let $w = \sum_{i=1}^{t} f_i \epsilon_i$. Let $\text{in}(f_i) = h_i$ for all $i$. The theorem is proved once we show that $\text{in}_\gamma(w) \in \langle \text{in}_\gamma(v) : v \in W \rangle$. Note that $\text{in}_\gamma(w) = \text{in}_\gamma(f_j \epsilon_j) = h_j \epsilon_j$ for some $j$. Let

$$\sigma = \sum_{i : h_i \text{in}(g_i) \equiv h_j \text{in}(g_j)} f_i \epsilon_i.$$

$\sigma$ is a syzygy on $\{\text{in}(g_i) : i \geq j\}$, because if $h_i \text{in}(g_i) \equiv_\gamma h_j \text{in}(g_j)$, we must have $i \geq j$. Hence, by Lemma 2.2.1, $\sigma$ is generated by $\sigma_{uv}$ for $u, v \geq j$, and $\epsilon_j$ only appears in $\sigma_{jv}$ for $j < v$. This implies that $h_j$ is a $\mathsf{k}$-linear combination of $\{m_{vj} : j < v\}$ and thus, $\text{in}_\gamma(w)$ is a $\mathsf{k}$-linear combination of $\{m_{vj}\epsilon_j : j < v\}$, which proves the theorem. $\square$

**Corollary 2.4.5.** *With notation as in Theorem 2.4.4, suppose that the $g_i$ are arranged such that whenever $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis vector $e$ of $F$, say $\text{in}(g_i) = m_i e, \text{in}(g_j) = m_j e$ with $m_i, m_j \in S$,*

$$i < j \implies m_i > m_j \text{ in lexicographic order.}$$

*If the variables $x_1, \dots, x_s$ are missing from $\text{in}(g_i)$ for all $i$, then the variables $x_1, \dots, x_{s+1}$ are missing from $\text{in}_\gamma(w_{ij})$ for all $i < j$ for which $w_{ij}$ is defined. Further, $F/\langle g_1, \dots, g_t \rangle$ has a free resolution of length $\leq r - s$.*

*Proof.* If the variables $x_1, \dots, x_s$ are missing from $\text{in}(g_i)$ for all $i$, then, due to the stipulated arrangement of $\{g_1, \dots, g_t\}$, for $i < j$ such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element, the variable $x_{s+1}$ must appear in $g_i$ with at least as high a power as in $g_j$. As a result, the variable $x_{s+1}$ does not appear in $m_{ji}$, and hence, does not appear in $\text{in}_\gamma(w_{ij}) = m_{ji}\epsilon_i$.
We now show that $F/(g_1, \dots, g_t)$ has a free resolution of length $\leq r - s$ by induction on $r - s$. Suppose first that $r - s = 0$, so that none of the variables $x_1, \dots, x_r$ appears in the terms $\text{in}(g_i)$. Since none of the variables appear in $\text{in}(g_i)$ for all $i$, $\text{in}(g_i)$ must be a scalar times a basis element of $F$. Let $F'$ be the free submodule spanned by all the $e_j$ which do not appear in $\text{in}(g_i)$ for any $i$. By Theorem 2.3.4, $F'$ is isomorphic to $F/(g_1, \dots, g_t)$.
Suppose $r - s > 0$. By the first statement of the theorem, the variables $x_1, \dots, x_{s+1}$ are missing from $\text{in}_\gamma(w_{ij})$ for all $i, j$. Order the $w_{ij}$ to satisfy the same hypothesis as on the $g_i$. Then, by the induction hypothesis, $F/\langle W \rangle$ has a free resolution of length $\leq r - s - 1$. Combining this with the natural map $\phi : \oplus S\epsilon_i \to F$, we get a free resolution of $F/\langle g_1, \dots, g_t \rangle$ of length $\leq r - s$. $\square$

**Example 2.4.6.** Let $F = S$ and $I = \langle x^3 - yz, y^2 - xz, x^2y - z^2 \rangle$. Let $g_1 = x^3 - yz, g_2 = y^2 - xz, g_3 = x^2y - z^2$. In this example, we consider the lexicographic order on $S$. Thus, we have

$$\text{in}(g_1) = x^3, \text{in}(g_2) = -xz, \text{in}(g_3) = x^2y.$$

Let $S_{ij} = m_{ji}g_i = m_{ij}g_j$. Then,

$$\begin{aligned} S_{12} &= \frac{-xz}{x}(x^3 - yz) - \frac{x^3}{x}(y^2 - xz) \\ &= yz^2 - x^2y^2 \\ &= -yg_3, \end{aligned}$$

and hence, $h_{12} = 0$. Similarly, $S_{23} = xy^3 - z^3 = h_{23}$. Thus, we add $g_4 = h_{23}$ to the original basis $\{g_1, g_2, g_3\}$. For the basis $\{g_1, g_2, g_3, g_4\}$, we immediately have $h_{12} = h_{23} = 0$. Calculation also reveals that $S_{13} = -zg_2$ and $S_{14} = -z(y^2 + xz)g_2$, which implies that $h_{13} = h_{14} = 0$. However, $S_{24} = y^5 - z^4 = h_{24}$. For the new basis $\{g_1, g_2, g_3, g_4, g_5\}$, where $g_5 = y^5 - z^4$, we instantly have $h_{12} = h_{23} = h_{13} = h_{14} = h_{24} = 0$. Further,

$$S_{34} = -z^2 g_2, S_{15} = -z(y^4 + xy^2 z + x^2 z^2)g_2, S_{25} = z^4 g_2 + y^2 g_5, S_{35} = -z^2(y^2 + xz)g_2, S_{45} = -z^3 g_2.$$

This shows that $\{g_1, g_2, g_3, g_4, g_5\}$ is a Gröbner basis of $I$.
Rearranging the basis to satisfy the hypothesis of the corollary, we have $I = \langle x^3 - yz, x^2y - z^2, xy^3 - z^3, xz - y^2, y^5 - z^4 \rangle$. Hence,

$$\begin{aligned}
w_{12} &= y\epsilon_1 - x\epsilon_2 - z\epsilon_4 \\
w_{13} &= y^3\epsilon_1 - x^2\epsilon_2 - z\epsilon_4 \\
w_{14} &= z\epsilon_1 - x^2\epsilon_4 - z(y^2 + xz)\epsilon_4 \\
w_{15} &= y^5\epsilon_1 - x^3\epsilon_5 - z(y^4 + xy^2z + x^2z^2)\epsilon_4 \\
w_{23} &= y^2\epsilon_2 - x\epsilon_3 - z^2\epsilon_4 \\
w_{24} &= z\epsilon_2 - xy\epsilon_4 - \epsilon_3 \\
w_{25} &= y^4\epsilon_2 - x^2\epsilon_5 - z^2(y^2 + xz)\epsilon_4 \\
w_{34} &= z\epsilon_3 - y^3\epsilon_4 + \epsilon_5 \\
w_{35} &= y^2\epsilon_3 - x\epsilon_5 - z^3\epsilon_4 \\
w_{45} &= (y^5 - z^4)\epsilon_2 + (y^2 - xz)\epsilon_5
\end{aligned}$$

Note that $x$ is missing from the initial terms of all the $w_{ij}$, as it should be, according to the previous corollary.

As a corollary, we get the following famous theorem by Hilbert.

**Theorem 2.4.7** (Hilbert's Syzygy Theorem)**.** *Let $M$ be a finitely generated $S$-module, where $S = \mathsf{k}[x_1, \ldots, x_r]$. Then,* $\operatorname{pdim}(M) \le r$.

# Chapter 3

# Ideals and related objects

## 3.1 Homological invariants of initial ideals

### 3.1.1 Gradings defined by weights

**Definition 3.1.1.** *Let* $\mathbf{w} = (w_1, \ldots, w_r) \in \mathbb{N}^r$*. We call this vector a* ***weight*** *and set* $\deg_{\mathbf{w}} x_i = w_i$ *for* $i = 1, \ldots, n$*. Then, for* $(a_1, \ldots, a_r) \in \mathbb{N}^r$*,*

$$\deg_{\mathbf{w}} x_1^{a_1} \ldots x_r^{a_r} = \sum_{i=1}^{r} a_i w_i.$$

*A polynomial* $f \in S$ *is called* ***homogeneous of degree*** $j$ *with respect to the weight* $\mathbf{w}$ *if the degree of all homogeneous components of* $f$ *is* $j$*.*

Fix a weight $\mathbf{w}$ and let $S_j$ be the $\mathsf{k}$-vector space spanned by all homogeneous polynomials of degree $j$. Then, $S_j$ is finite dimensional and the monomials $u$ with $\deg_{\mathbf{w}} u = j$ form a $\mathsf{k}$-basis. It follows that

$$S = \oplus_j S_j.$$

Thus, note that we have defined a new grading on $S$.

**Definition 3.1.2.** *Each polynomial* $f \in S$ *can be uniquely written as* $f = \sum_j f_j$ *with* $f_j \in S_j$*. The summands* $f_j$ *are called the* ***homogeneous components*** *of* $f$ *with respect to* $\mathbf{w}$*.*
*The* ***degree*** *of* $f$ *with respect to* $\mathbf{w}$ *is defined to be* $\deg_{\mathbf{w}} f = \max\{j : f_j \neq 0\}$*, and if* $i = \deg_{\mathbf{w}} f$*, then* $f_i$ *is called the* ***initial term*** *of* $f$ *with respect to* $\mathbf{w}$ *and is denoted by* $\mathrm{in}_{\mathbf{w}}(f)$*.*

Note that $\mathrm{in}_{\mathbf{w}}(f)$ need not be a monomial.

**Definition 3.1.3.** *Let* $I \subset S$ *be an ideal. We define the* ***initial ideal*** *of* $I$ *with respect to* $\mathbf{w}$ *as*

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) : f \in I \rangle.$$

*A set of polynomials* $f_1, \ldots, f_n \in I$ *such that* $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \ldots, \mathrm{in}_{\mathbf{w}}(f_n) \rangle$ *is called a standard basis of* $I$ *with respect to* $\mathbf{w}$*.*

The following lemma shows that a standard basis of $I$ with respect to a weight generates $I$.

**Lemma 3.1.4.** *Let $J \subset I$ be ideals in $S$. If $\text{in}_{\mathbf{w}}(J) = \text{in}_{\mathbf{w}}(I)$, then $I = J$.*

*Proof.* Suppose $I \neq J$. Let $f \in I \backslash J$ such that $\deg_{\mathbf{w}} f$ is minimum among all elements in $I \backslash J$. Since $\text{in}_{\mathbf{w}}(f) \in \text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{w}}(J)$ and $\text{in}_{\mathbf{w}}(J)$ is a homogeneous ideal with respect to the grading given by $\mathbf{w}$, there must exist $g \in J$ such that $\text{in}_{\mathbf{w}}(f) = \text{in}_{\mathbf{w}}(g)$. Note that $f - g \in I \backslash J$, and $\deg_{\mathbf{w}}(f - g) < \deg_{\mathbf{w}}(f)$, which is a contradiction. □

The following lemma is proved in [12].

**Lemma 3.1.5.** *Given a monomial order $\tau$ and pairs of monomials $(g_1, h_1), \ldots, (g_m, h_m)$ such that $g_i >_\tau h_i$ for all $i$, there exists a weight $\mathbf{w}$ such that $\deg_{\mathbf{w}} g_i > \deg_{\mathbf{w}} h_i$ for all $i$.*

**Theorem 3.1.6.** *Given an ideal $I$ and a monomial order $\tau$, there exists a weight $\mathbf{w}$ such that $\text{in}_\tau(I) = \text{in}_{\mathbf{w}}(I)$.*

*Proof.* Let $\{g_1, \ldots, g_n\}$ be a Gröbner basis of $I$ with respect to the monomial order $\tau$. For all $i$, define $K_i$ to be the set of all monomials appearing in $g_i$, and denote the monomial corresponding to $\text{in}_\tau(g_i)$ as $m_i$. Define $K = \sqcup_i (g_i, K_i \backslash \{m_i\}) \in S^2$. By the previous lemma, there exists a weight $\mathbf{w}$ such that $g > h$ for all $(g, h) \in K$. Observe that $\text{in}_{\mathbf{w}}(g_i) = \text{in}_\tau(g_i)$ for all $I$. Hence,

$$\text{in}_\tau(I) = \langle \text{in}_\tau(g_1), \ldots, \text{in}_\tau(g_n) \rangle \subset \text{in}_{\mathbf{w}}(I).$$

Define a monomial order $\tau_{\mathbf{w}}$ as $m_1 <_{\tau_{\mathbf{w}}} m_2$ if (i) $\deg_{\mathbf{w}}(m_1) < \deg_{\mathbf{w}}(m_2)$ or (ii) $\deg_{\mathbf{w}}(m_1) = \deg_{\mathbf{w}}(m_2)$ and $m_1 <_\tau m_2$. Thus, we have

$$\text{in}_\tau(I) = \text{in}_\tau(\text{in}_\tau(I)) \subset \text{in}_\tau(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\tau_{\mathbf{w}}}(I).$$

Corollary 2.3.6 implies that $\text{in}_\tau(I) = \text{in}_{\tau_{\mathbf{w}}}(I)$. We show that $\text{in}_{\tau_{\mathbf{w}}}(I) \supset \text{in}_{\mathbf{w}}(I)$ to complete the proof.
Observe that $\text{in}_{\tau_{\mathbf{w}}}(g_i) = \text{in}_\tau(g_i) = \text{in}_{\mathbf{w}}(g_i)$ for all $i$ and hence, $\{g_1, \ldots, g_n\}$ is a Gröbner basis of $I$ with respect to $\tau_{\mathbf{w}}$ as well.
Let $f \in I$ and $f = f_1 g_1 + \cdots + f_n g_n$ be a standard expression for $f$ in terms of $\{g_1, \ldots, g_n\}$. Since $\text{in}_{\tau_{\mathbf{w}}}(f) \geq_{\tau_{\mathbf{w}}} \text{in}_{\tau_{\mathbf{w}}}(f_i g_i)$ for all $i$, we have $\deg_{\mathbf{w}} f \geq \deg_{\mathbf{w}}(f_i g_i)$. Let $L = \{i \in \{1, \ldots, n\} : \deg_{\mathbf{w}} f = \deg_{\mathbf{w}}(f_i g_i)\}$. Then,

$$\text{in}_{\mathbf{w}}(f) = \sum_{i \in L} \text{in}_{\mathbf{w}}(f_i g_i) = \sum_{i \in L} \text{in}_{\mathbf{w}}(f_i) \text{in}_{\mathbf{w}}(g_i) = \sum_{i \in L} \text{in}_{\mathbf{w}}(f_i) \text{in}_{\tau_{\mathbf{w}}}(g_i) \in \text{in}_{\tau_{\mathbf{w}}}(I).$$

□

### 3.1.2 Homogenization

**Definition 3.1.7.** *Fix a weight* $\mathbf{w}$*. Let* $f$ *be a non-zero polynomial in* $S$ *with homogeneous components* $f_j$ *(with respect to the weight* $\mathbf{w}$*). We introduce a new variable* $t$ *and define the* ***homogenization*** *of* $f$ *with respect to* $\mathbf{w}$ *as the polynomial*

$$f^h = \sum_j f_j t^{\deg_{\mathbf{w}} f - j} \in S[t].$$

Note that $f^h$ is homogeneous in $S[t]$ with respect to the extended weight $(w_1, \ldots, w_r, 1) \in \mathbb{N}^{r+1}$.

**Definition 3.1.8.** *Let* $I \subset S$ *be an ideal. The* ***homogenization*** *of* $I$ *is defined to be the ideal*

$$I^h = \langle f^h : f \in I \rangle \subset S[t].$$

For any homogeneous polynomial $g \in S[t]$, let $\overline{g}$ denote the polynomial in $S$ obtained by substituting $t = 1$.

**Lemma 3.1.9.** *Let* $f \in S[t]$ *be homogeneous with respect to the weight* $(w_1, \ldots, w_r, 1)$*. Then* $f \in I^h$ *iff* $f = t^n g^h$ *for some* $g \in I$ *and some* $n \in \mathbb{Z}_{\geq 0}$*. Further, in this case,* $g = \overline{f}^h$*.*

*Proof.* It is clear that $f \in I^h$ if $f = t^n g^h$ for some $g \in I$ and some $n \in \mathbb{Z}_+$.
Suppose $f \in I^h$ is homogeneous. Then, there exist $f_1, \ldots, f_s \in I$ and $g_1, \ldots, g_s \in S[t]$ such that $f = \sum_{i=1}^s g_i f_i^h$.
We have

$$\overline{f} = \sum_{i=1}^{s} \overline{g_i}\overline{f_i^h} = \sum_{i=1}^{s} \overline{g_i} f_i \in I.$$

We claim that $f = t^n \overline{f}^h$ for some non-negative integer $n$. To observe this, let $f = g_l(x_1, \ldots, x_r)t^l + \cdots + g_k(x_1, \ldots, x_r)t^k$ such that $l \leq k$ and $g_l, g_k \neq 0$. Then, $\overline{f} = g_l(x_1, \ldots, x_r) + \cdots + g_k(x_1, \ldots, x_r)$ and

$$\overline{f}^h = g_l(x_1, \ldots, x_r) + g_{l+1}(x_1, \ldots, x_r)t + \cdots + g_k(x_1, \ldots, x_r)t^{k-l},$$

which implies that $f = t^l \overline{f}^h$ and completes the proof. □

**Remark 3.1.10.** *Observe that in the above proof, we have also shown that if* $f$ *is homogeneous in* $I^h$*, then* $\overline{f} \in I$*.*

**Definition 3.1.11.** *A monomial order* $\tau$ *on* $S$ *is said to respect* $\mathbf{w}$ *if for all* $m_1, m_2 \in S$ *such that* $\deg_{\mathbf{w}} m_1 < \deg_{\mathbf{w}} m_2$*, we have* $m_1 <_\tau m_2$*.*

**Example 3.1.12.** *The graded lexicographic order and reverse graded lexicographic order respect the standard grading on* $S$*. More generally, the order* $<_{\mathbf{w}}$ *respects* $\mathbf{w}$*.*

For a monomial order $\tau$ which respects $\mathbf{w}$, define a natural extension $\tau'$ to $S[t]$ as follows: $x^a t^c <_{\tau'} x^b t^d$ iff (i) $x^a <_\tau x^b$ or (ii) $x^a = x^b$ and $c < d$, where, as usual, $x^a$ denotes $x_1^{a_1} \ldots x_r^{a_r}$.
This monomial order has the property that $\mathrm{in}_\tau(g) = \mathrm{in}_{\tau'}(g^h)$ for all non-zero $g \in S$.

**Proposition 3.1.13.** *Let $I \subset S$ be an ideal, and let $\{g_1, \dots, g_n\}$ be a Gröbner basis of $I$ with respect to a monomial order $\tau$ which respects $\mathbf{w}$. Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of $I^h$ with respect to $\tau'$.*

*Proof.* Note that since $I^h$ is a homogeneous ideal with respect to the extended weight $(w_1, \dots, w_r, 1)$, it is sufficient to prove that if $f \in I^h$ is homogeneous with respect to $(w_1, \dots, w_r, 1)$, then $\text{in}_{\tau'}(f) \in \langle \text{in}_{\tau'}(g_1^h), \dots, \text{in}_{\tau'}(g_n^h) \rangle$.
Let $f \in I^h$, be homogeneous. Then, by the previous lemma, there exist $g \in I$ and $m \in \mathbb{Z}_+$ such that $f = t^m g^h$. Hence,

$$\text{in}_{\tau'}(f) = t^m \text{in}_{\tau'}(g^h) = t^m \text{in}_\tau(g).$$

There exist $u \in S$ and $i \in \{1, \dots, n\}$ such that $\text{in}_\tau(g) = u\text{in}_\tau(g_i) = u\text{in}_{\tau'}(g_i^h)$. Thus, $\text{in}_{\tau'} f = ut^m \text{in}_{\tau'}(g_i^h)$. □

**Proposition 3.1.14.** *Given an ideal $I \subset S$, $S[t]/I^h$ is a free $\mathsf{k}[t]$-module.*

*Proof.* Let $\{g_1, \dots, g_n\}$ be a Gröbner basis of $I$ with respect to a monomial order $\tau$ graded with respect to $\mathbf{w}$. Then, $\{g_1^h, \dots, g_n^h\}$ is a Gröbner basis of $I^h$ with respect to $\tau'$. It follows from Theorem 2.3.4 that the set of all monomials in $S[t]$ not in $\langle \text{in}_{\tau'}(g_1^h), \dots, \text{in}_{\tau'}(g_n^h) \rangle$ forms a $\mathsf{k}$-basis of $S[t]/I^h$. Since $\text{in}_{\tau'}(g_i^h) = \text{in}_\tau(g_i)$, we have $\langle \text{in}_{\tau'}(g_1^h), \dots, \text{in}_{\tau'}(g_n^h) \rangle = \langle \text{in}_\tau(g_1), \dots, \text{in}_\tau(g_n) \rangle S[t]$ and hence, the set of all monomials in $S$ not in $\langle \text{in}_\tau(g_1), \dots, \text{in}_\tau(g_n) \rangle$ forms a $\mathsf{k}[t]$-basis of $S[t]/I^h$. □

**Lemma 3.1.15.** *Let $R$ be a ring and consider $\phi : R[t] \to R$, a ring homomorphism with $\phi|_R = Id$, or equivalently, an $R$-linear ring homomorphism. Given an ideal $I \in R[t]$, $\phi$ naturally induces an $R$-linear ring homomorphism $\overline{\phi} : R[t]/I \to R/\phi(I)$ given by $\overline{\phi}(\overline{f}) = \overline{\phi(f)}$, and $\ker(\overline{\phi}) = (t - \phi(t))R[t]/I$.*

*Proof.* Clearly, $\overline{\phi}(f)$ is well-defined and $(t - \phi(t))R[t]/I \subset \ker(\overline{\phi})$.
Let $f \in R[t]$ such that $\overline{f} \in \ker(\overline{\phi})$. There exist $a \in R$ and $g \in R[t]$ such that $f = a + (t - \phi(t))g$, which implies that $\overline{\phi}(\overline{f}) = \overline{a}$. Thus, we have $a \in \phi(I)$. Let $h \in I$ such that $\phi(h) = a$, that is, $h = a + (t - \phi(t))h'$. Then,

$$f - h \in (t - \phi(t))R[t] \implies f \in I + (t - \phi(t))R[t],$$

which completes the proof. □

**Proposition 3.1.16.** *Given an ideal $I \subset S$ and a weight $\mathbf{w}$ on $S$, we have the following $S$-linear ring isomorphisms:*

$$\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\text{in}_\mathbf{w}(I) \text{ and } \frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/I \ \forall a \in S \setminus \{0\}.$$

*Proof.* For all $a \in \mathsf{k}$, define an $S$-linear map $\phi_a : S[t] \to S$ as $\phi_a(1) = 1$ and $\phi_a(t) = a$. We claim that $\phi_0(I^h) = \text{in}_\mathbf{w}(I)$.

Given $f \in I$, $\phi_0(f^h) = \text{in}_{\mathbf{w}}(f)$. Since $I^h = \langle f^h : f \in I \rangle$, it follows that $\phi_0(I^h) = \text{in}_{\mathbf{w}}(I)$. From the previous lemma, we have $\frac{S[t]/I^h}{tS[t]/I^h} \cong S\text{in}_{\mathbf{w}}(I)$.
For $a \neq 0$, define a ring homomorphism $\psi_a : S \to S$ as $\psi_a(x_i) = a^{w_i}x_i$ for all $i$ and $\psi_a|_{\mathsf{k}} = Id$. We claim that $\psi_a\phi_a(I^h) = I$. Then, according to the previous lemma, $\frac{S[t]/I^h}{(t-a)S[t]/I^h} \cong S/\phi_a(I^h)$ as $S$-modules and since $a \neq 0$, $\psi_a$ is a ring isomorphism and $S/\phi_a(I^h) \cong S/I$ as rings.
By Proposition 3.1.13, there exists a Gröbner basis $\{g_1, \ldots, g_n\}$ of $I$ such that $\{g_1^h, \ldots, g_n^h\}$ is a Gröbner basis of $I^h$. Let $g_i = \sum_j g_{ij}$ where $g_{ij}$ denotes the homogeneous component of $g_i$ of degree $j$ (with respect to $\mathbf{w}$). Then,

$$\phi_a(g_i^h) = \sum_j a^{\deg_{\mathbf{w}} g_i - j} g_{ij},$$

and

$$\psi_a(\phi_a(g_i^h)) = a^{\deg_{\mathbf{w}} g_i} g_i.$$

Since $a \neq 0$, we are done. □

We now compare the Betti numbers of an ideal with those of its initial ideal.
Let $I \subset S$ be a graded ideal with respect to the standard grading on $S$, and fix a weight $\mathbf{w}$ on $S$. Let $\{g_1, \ldots, g_n\}$ be a Gröbner basis of $I$ with respect to a monomial order which respects $\mathbf{w}$, and further, such that $g_i$ is homogeneous with respect to the standard grading for all $i$. Then, $\{g_1^h, \ldots, g_n^h\}$ is a system of generators (in fact, a Gröbner basis) of $I^h$.
If we assign to each $x_i$ the bidegree $(w_i, 1)$ and to $t$ the bidegree $(1, 0)$, then all the generators $g_i^h$ are bihomogeneous, and hence $I^h$ is a bigraded ideal. Therefore $S[t]/I^h$ has a bigraded minimal free $S[t]$-resolution,

$$F_\bullet : 0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to S[t]/I^h \to 0,$$

where $F_i = \oplus_{j,k}(S[t](-k, -j))^{\beta_{ijk}}$. Note that the minimality of the resolution is equivalent to the condition that all entries in the matrices describing the maps must belong to $\langle x_1, \ldots, x_r, t \rangle$.
Note that as $S[t]/I^h$ is a free $\mathsf{k}[t]$-module, $t - a$ is a non-zero divisor on $S[t]/I^h$ for all $a \in \mathsf{k}$. Since $t$ is a non-zero divisor on $S[t]/I^h$ and on $S[t]$, and $t \in \langle x_1, \ldots, x_r, t \rangle$, $F_\bullet/tF_\bullet$ is a bigraded minimal free $S$-resolution of $\frac{S[t]/I^h}{tS[t]/I^h} \cong S/\text{in}_{\mathbf{w}}(I)$. Observe that the bigraded shifts of $F_\bullet/tF_\bullet$ are the same as those in $F_\bullet$ and in particular, the second component of the shifts in the resolution are the ordinary shifts of the standard graded ideal $\text{in}_{\mathbf{w}}(I)$. Thus, we have

$$\beta_{ij}(S/\text{in}_{\mathbf{w}}(I)) = \sum_k \beta_{ijk} \textit{ for all } i, j.$$

On the other hand, since $t - 1$ is also a non-zero divisor on $S[t]/I^h$ and on $S[t]$, $F_\bullet/(t-1)F_\bullet$ is a free $S$-resolution of $\frac{S[t]/I^h}{(t-1)S[t]/I^h} \cong S/I$. Note that $t - 1$ is homogeneous with respect to the second component of the bidegree and hence the second components of the shifts in the resolution $F_\bullet$ are preserved. However, $t - 1$ does not belong to $\langle x_1, \ldots, x_r, t \rangle$ and hence $F_\bullet/(t-1)F_\bullet$ need not be a minimal resolution. Therefore, we have

$$\beta_{ij}(S/I) \leq \sum_k \beta_{ijk} \textit{ for all } i, j.$$

We have thus proved the following theorem.

**Theorem 3.1.17.** *Let $I \subset S$ be a graded ideal and $\mathbf{w}$ be a weight. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(\mathrm{in}_{\mathbf{w}}(I)) \textit{ for all } i,j.$$

Theorem 3.1.17 and Thereom 3.1.6 yield the following corollary.

**Corollary 3.1.18.** *Let $I \subset S$ be a graded ideal and $\tau$ be a monomial order on $S$. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(\mathrm{in}_{\tau}(I)) \textit{ for all } i,j.$$

Once the Castelnuovo-Mumford regularity is introduced later, the following result follows immediately from Corollary 3.1.18.

**Corollary 3.1.19.** *Let $I \subset S$ be a graded ideal and $\tau$ be a monomial order on $S$. Then* $\mathrm{reg}(I) \leq \mathrm{reg}(\mathrm{in}(I))$.

**Corollary 3.1.20.** *Given $I, \tau$ as above,*
*(i)* $\mathrm{pdim}(S/I) \leq \mathrm{pdim}(S/\mathrm{in}_{\tau}(I))$*;*
*(ii)* $\mathrm{depth}(S/I) \geq \mathrm{depth}(S/\mathrm{in}_{\tau}(I))$.

*Proof.* Corollary 3.1.18 directly implies (a). (b) follows from (a) and the Auslander-Buchsbaum formula. □

**Proposition 3.1.21.** *Let $I \subset S$ be a graded ideal. Then,*
*(i) If* $\mathrm{in}_{\mathbf{w}}(I)$ *is a prime ideal, so is $I$.*
*(ii) If* $\mathrm{in}_{\mathbf{w}}(I)$ *is a radical ideal, so is $I$.*

*Proof.* Let $I^h \in S[t]$ be the homogenization of $I$ with respect to the weight $\mathbf{w}$. We claim that $I$ is prime (resp. radical) if $I^h$ is prime (resp. radical).
$\phi(f^h) = t^{\deg_{\mathbf{w}} f} f$.
Suppose $I^h$ is prime. Consider $f, g \in S \backslash \{0\}$ such that $fg \in I$. Then, $(fg)^h = f^h g^h \in I^h$, which implies that $f^h \in I^h$ or $g^h \in I^h$. Without loss of generality, let $f^h \in I^h$. Then, by Remark 3.1.10, note that $f = \overline{(f^h)} \in I$.
Similarly, suppose $I^h$ is radical. Consider $f \in S\backslash\{0\}$ such that $f^n \in I$ for $n \in \mathbb{N}$. Then, $(f^n)^h = (f^h)^n \in I^h$ and hence, $f^h \in I^h$. Proceeding as above, we have $f \in I$.
The following lemma along with Proposition 3.1.16 proves that if $\mathrm{in}_{\mathbf{w}}(I)$ is prime (resp. radical), so is $I^h$. □

**Lemma 3.1.22.** *Let $R$ be a finitely generated positively graded* $\mathsf{k}$*-algebra and let $s \in R$ be a homogeneous non-zero divisor of $R$ such that $R/sR$ is a domain (resp. a reduced ring) and* $\deg(s) > 0$*. Then $R$ is also a domain (resp. a reduced ring).*

*Proof.* Suppose $R/sR$ is a domain and there exist $a, b \in R\backslash\{0\}$ such that $ab = 0$. By the Krull Intersection Theorem, $\cap_{k \geq 0} \langle s \rangle^k = 0$ and hence, there exist $n_a, n_b \in \mathbb{Z}_{\geq 0}$ such that $a \in \langle s \rangle^{n_a}, b \in \langle s \rangle^{n_b}$ and $a \notin \langle s \rangle^{n_a+1}, b \notin \langle s \rangle^{n_b+1}$. Let $a = a' s^{n_a}, b = b' s^{n_b}$ where $a', b' \notin \langle s \rangle$. Then, $a'b' = 0$ and hence $\overline{a'b'} = 0$, which implies that $a' \in \langle s \rangle$ or $b' \in \langle s \rangle$, a contradiction.
Similarly, suppose $R/sR$ is a reduced ring and there exists $a \in R\backslash\{0\}$ such that $a^n = 0$. Let $n_a$ and $a'$ be as above. Then, $a'^n = 0$ and hence $\overline{a'}^n = 0$, which implies that $a' \in \langle s \rangle$, a contradiction. □

## 3.2 Polarization

As usual, let $S = \mathsf{k}[x_1, \dots, x_r]$.

**Lemma 3.2.1.** *Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \dots, m_n\}$, where $m_i = \prod_{j=1}^{n} x_j^{a_{ij}}$ for $i = 1, \dots, n$. Fix an integer $j \in [n]$ and suppose that $a_{ij} > 1$ for at least one $i \in [r]$. Let $T = S[y]$ and let $J \subset T$ be the monomial ideal with minimal generating set of monomials $\{m'_1, \dots, m'_n\}$, where*

$$m'_i = \begin{cases} m_i & a_{ij} = 0 \\ (m_i/x_j)y & a_{ij} \geq 1. \end{cases}$$

*Then $y - x_j$ is a non-zero divisor in $T/J$ and*

$$\frac{T/J}{(y - x_j)T/J} \cong S/I$$

*as $S$-modules.*

*Proof.* Suppose $y - x_j$ is a zero divisor in $T/J$. Then $y - x_j \in P$ for some $P \in \mathrm{Ass}(J)$. By applying Proposition 1.1.5 on the $\mathbb{N}^r$-grading, $P$ is a monomial ideal, and hence $y, x_j \in P$. Thus, there exists a monomial $f \in T \backslash J$ such that $yf, x_j f \in J$. Then there exist $m'_k, m'_l$ and monomials $f_1, f_2 \in T$ such that $yf = m'_k f_1$ and $x_j f = m'_l f_2$.
Since $f \notin J$, $x_j$ divides $m'_l$ and hence, by the construction of $J$, $y$ divides $m'_l$. This implies that $y$ divides $f$. Note that $y$ does not divide $f_1$ because $f \notin J$. This forces $y^2$ to divide $m'_k$, which is a contradiction to the construction of $J$.
Define a ring homomorphism $\phi : T \to S$ such that $\phi|_S = Id$ and $\phi(y) = x_j$. Then, $\phi(J) = I$ and by Lemma 3.1.15, we have the required isomorphism. □

Motivated by Lemma 3.2.1, we define the polarization of a monomial ideal $I$.
Let $I \subset S$ be a monomial ideal with minimal generating set of monomials $\{m_1, \dots, m_n\}$, where $m_i = \prod_{j=1}^{n} x_j^{a_{ij}}$ for $i = 1, \dots, n$. For all $j = 1, \dots, r$, define $a_j = \max\{a_{ij} : i = 1, \dots, n\}$.
Let $T = \mathsf{k}[x_{11}, x_{12} \dots, x_{1a_1}, x_{21}, \dots, x_{2a_2}, \dots, x_{n1}, \dots, x_{na_n}]$. Define $J$ to be a monomial ideal in $T$ with generating set $\{m'_1, \dots, m'_n\}$ where

$$m'_i = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk}$$

for all $i \in [n]$.

**Definition 3.2.2.** *The monomial ideal $J$ is called the **polarization** of $I$.*

**Example 3.2.3.** *Consider the ideal $\langle x_1 x_2^2, x_2^4 \rangle \subset \mathsf{k}[x_1, x_2]$. The polarisation of $I$ is*

$$J = \langle x_{11}, x_{21}x_{22}, x_{21}x_{22}x_{23}x_{24} \rangle \subset \mathsf{k}[x_{11}, x_{21}, x_{22}, x_{23}, x_{24}].$$

**Proposition 3.2.4.** *Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then the sequence* $\mathbf{z}$ *given by*

$$x_{n1} - x_{na_n}, \ldots, x_{n1} - x_{n2}, \ldots, x_{21} - x_{2a_2}, \ldots, x_{21} - x_{22}, \ldots, x_{11} - x_{1a_1}, \ldots, x_{11} - x_{12}$$

*is a regular sequence on $T/J$ and*

$$\frac{T/J}{(z)T/J} \cong S/I$$

*as graded* $\mathsf{k}$*-algebras.*

*Proof.* Firstly, replace $x_i$ in $S$ by $x_{i1}$ for all $i \in [r]$. Let the minimal generating set of monomials of $I$ be $\{m_1^{(11)}, \ldots, m_n^{(11)}\}$ Now, let $T_{12} = S[x_{12}]$ and define $m_i^{(12)} = m_i^{(11)}$ if $x_{11}$ does not appear in $m_i^{(11)}$ and $m_i^{(12)} = (m_i^{(11)}/x_{11})x_{12}$ otherwise. Let $J_{12} = \langle m_1^{(12)}, \ldots, m_n^{(12)} \rangle$. By Lemma 3.2.1, $x_{11} - x_{12}$ is a non-zero divisor on $T_{12}/J_{12}$ and

$$\frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Similarly, let $T_{13} = T_{12}[x_{13}]$ and define $J_{13} = \langle m_1^{(13)}, \ldots, m_n^{(13)} \rangle$ $m_i^{(13)} = m_i^{(11)}$ if $x_{11}$ does not appear in $m_i^{(11)}$ and $m_i^{(13)} = (m_i^{(11)}/x_{11})x_{13}$ otherwise. Note that

$$\frac{T_{13}/J_{13}}{(x_{11} - x_{13}, x_{11} - x_{12})T_{13}/J_{13}} \cong \frac{T_{12}/J_{12}}{(x_{11} - x_{12})T_{12}/J_{12}} \cong S/I.$$

Continue the process until $T_{1a_1}$. Then, let $T_{22} = T_{1a_1}[x_{22}]$. We eventually get $T_{na_n} = T$. Repeated application of Lemma 3.2.1 completes the proof. □

**Corollary 3.2.5.** *Let $I \subset S$ be a monomial ideal and $J \subset T$ be its polarization. Then*
*(i)* $\beta_{ij}(I) = \beta_{ij}(J)$ *for all* $i, j$*;*
*(ii)* $H_{S/I}(t) = (1-t)^{\delta} H_{T/J}(t)$ *where* $\delta = \dim T - \dim S$*;*
*(iii)* $\operatorname{pdim}(S/I) = \operatorname{pdim}(S/J)$ *and* $\operatorname{reg}(S/I) = \operatorname{reg}(T/J)$.

*Proof.* (i) Follows from the fact that $\mathbf{z}$ is a regular sequence on $T/J$.
(ii) Follows from Corollary 1.2.9.
(iii) Follows from (i). □

## 3.3 The lexsegment ideal

Given a graded ideal $I \subset S$, our aim is to show the existence of a special ideal, the lexsegment ideal of $I$, denoted by $I^{\text{lex}}$, such that $S/I$ and $S/I^{\text{lex}}$ have the same Hilbert function.
By Corollary 2.3.5, $S/I$ and $S/\text{in}_\tau(I)$ have the same Hilbert function for any monomial order $\tau$ on $S$. Thus, we can assume that $I$ is a monomial ideal. By Theorem 2.3.4, the monomials in $S$

not belonging to $I$ form a $\mathsf{k}$-basis of $I$ and since this $\mathsf{k}$-basis determines the Hilbert functions of $S/I$, the Hilbert function of $S/I$ does not depend on the base field $\mathsf{k}$. We can therefore assume that $\text{char}(\mathsf{k}) = 0$.
We denote by $M_d(S)$ the set of all monomials of $S$ of degree $d$.

**Definition 3.3.1.** *A set $\mathcal{L} \subset M_d(S)$ is called a **lexsegment** if for all $m \in \mathcal{L}$, we have that $m' \in \mathcal{L}$ for all $m' \in M_d(S)$ such that $m' \geq_{\text{lex}} m$.*

**Definition 3.3.2.** *A set $\mathcal{L} \subset M_d(S)$ is called **strongly stable** if $x_i(m/x_j) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all pairs $(i,j)$ such that $i < j$ and $x_j$ divides $m$.*

For a monomial $m \in S$, we set $\gamma(m) = \max\{i : x_i \, divides \, m\}$.

**Definition 3.3.3.** *A set $\mathcal{L} \subset M_d(S)$ is called **stable** if $x_i(m/x_\gamma(m)) \in \mathcal{L}$ for all $m \in \mathcal{L}$ and all $i < \gamma(m)$.*

**Definition 3.3.4.** *A monomial ideal $I$ is said to be a lexsegment ideal or a (strongly) stable monomial ideal, if for each $d$ the monomials of degree $d$ in $I$ form a lexsegment, or a (strongly) stable set of monomials respectively.*

**Remark 3.3.5.** Note that every lexsegment set is strongly stable, and every strongly stable set is stable.

**Example 3.3.6.** *Let $S = \mathsf{k}[x,y,z,w]$.*
*Suppose $I_1$ is the smallest lexsegment ideal containing $xyz$. Then $I_1 = \langle xyz, xy^2, x^2w, x^2z, x^2y, x^3 \rangle$.*
*Suppose $I_2$ is the smallest strongly stable ideal containing $xyz$. Then $I_2 = \langle xyz, xy^2, x^2z, x^2y, x^3 \rangle$.*
*Suppose $I_3$ is the smallest stable ideal containing $xyz$. Then $I_3 = \langle xyz, xy^2, x^2y, x^3 \rangle$.*

Now we have that $S/I$ and $S/\text{gin}_\tau(I)$ have the same Hilbert function, and that $gin_\tau(I)$ is a strongly stable ideal [12]. Hence, we can assume that $I$ is a strongly stable ideal.

**Theorem 3.3.7.** *Let $I \subset S$ be a graded ideal. There exists a unique lexsegment ideal, denoted $I^{\text{lex}}$, such that $S/I$ and $S/I^{\text{lex}}$ have the same Hilbert function.*

Given a graded ideal $I$, with $j^{th}$ graded component $I_j$, denote by $I_j^{\text{lex}}$ the $\mathsf{k}$-vector space spanned by the unique lexsegment $\mathcal{L}_j$ with $|\mathcal{L}_j| = \dim_{\mathsf{k}} I_j$. Define $I^{\text{lex}} = \oplus_j I_j^{\text{lex}}$.
Note that if $I^{\text{lex}}$ as defined above is an ideal, it is the only possible lexsegment ideal such that $S/I$ and $S/I^{\text{lex}}$ have the same Hilbert function. Therefore, we only need to show that $I^{\text{lex}}$ is an ideal to prove Theorem 3.3.7. It is sufficient to show that $\{x_1, \ldots, x_r\}\mathcal{L}_j \subset \mathcal{L}_{j+1}$.

**Definition 3.3.8.** *Let $\mathcal{N}$ be a set of monomials in $S$. Then the shadow of $\mathcal{N}$ is said to be the set*

$$\text{Shad}(\mathcal{N}) = \{x_1, \ldots, x_r\}\mathcal{N} = \{x_i u : u \in N, i = 1, \ldots, n\}.$$

**Lemma 3.3.9.** *If $\mathcal{N} \subset M_d(S)$ is stable, strongly stable or lexsegment, then so is $\text{Shad}(\mathcal{N})$.*

Given $\mathcal{N} \subset M_d(S)$, we denote by $\gamma_i(N)$ the number of elements $\gamma(m) = i$ and set $\gamma_{\leq i}(\mathcal{N}) = \sum_{j=1}^{i} \gamma_i(\mathcal{N})$.

**Lemma 3.3.10.** *Let $\mathcal{N} \subset M_d(S)$ be a stable set of monomials. Then* $\mathrm{Shad}(N)$ *is a stable set and*
*(i)* $\gamma_i(\mathrm{Shad}(N)) = \gamma_{\leq i}(\mathcal{N})$*;*
*(ii)* $|\mathrm{Shad}(N)| = \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{N})$.

*Proof.* (ii) follows directly from (i). To prove (i), define the map

$$\phi : \{m \in \mathcal{N} : \gamma(m) \leq i\} \to \{m \in \mathrm{Shad}(\mathcal{N}) : \gamma(m) = i\},\ m \to mx_i.$$

$\phi$ is clearly injective. Let $m' \in \mathrm{Shad}(\mathcal{N})$ such that $\gamma(m') = i$. There exists $j \in [r]$ and $m \in \mathcal{N}$ such that $m' = x_j m$. We must have $\gamma(m) \leq i$. If $j = i$, then we are done. If $j < i$, then $\gamma(m) = i$ and since $\mathcal{N}$ is stable, $m_1 = x_j(m/x_i) \in \mathcal{N}$. Hence, we have $m' = x_i m_1$ for $m_1 \in \mathcal{N}$. This proves that $\phi$ is a bijection, which implies (i). □

**Theorem 3.3.11** (Bayer)**.** *Let $\mathcal{L} \subset M_d(S)$ be a lexsegment and $\mathcal{N} \subset M_d(S)$ be a strongly stable set of monomials with $|\mathcal{L}| \leq |\mathcal{N}|$. Then $\gamma_{\leq i}(\mathcal{L}) \leq \gamma_{\leq i}(\mathcal{N})$ for $i = 1, \ldots, r$.*

*Proof.* Observe that we can write $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1 x_r \cup \cdots \cup \mathcal{N}_d x_r^d$ where each $\mathcal{N}_j$ is a strongly stable set of monomials of degree $d - j$ in the variables $x_1, \ldots, x_{r-1}$. The lexsegment $\mathcal{L}$ has a similar decomposition $\mathcal{L}_0 \cup \cdots \cup \mathcal{L}_r x_r$, where each $\mathcal{L}_j$ is a lexsegment.
We prove the theorem by induction on the number of variables. If $r = 1$, we have that $\gamma_{\leq 1}(\mathcal{L}) = |\mathcal{L}| \leq |\mathcal{N}| = \gamma_{\leq 1}(\mathcal{N})$.
Let $r > 1$. We have that $\gamma_{\leq r}(\mathcal{L}) = |\mathcal{L}|$ and $\gamma_{\leq r}(\mathcal{N}) = |\mathcal{N}|$ and hence, $\gamma_{\leq r}(\mathcal{L}) \leq \gamma_{\leq r}(\mathcal{N})$. Note that for $i < r$, $\gamma_{\leq i}(\mathcal{L}) = \gamma_{\leq i}(\mathcal{L}_0)$ and $\gamma_{\leq i}(\mathcal{N}) = \gamma_{\leq i}(\mathcal{N}_0)$. Hence, if we show that $|\mathcal{L}_0| \leq |\mathcal{N}_0|$, the proof is done by induction.
For each $j$, let $\mathcal{N}_j^*$ be the lexsegment in $M_{d-j}(\mathsf{k}[x_1, \ldots, x_{r-1}])$ with $|\mathcal{N}_j^*| = |\mathcal{N}_j|$ and let $\mathcal{N}^* = \mathcal{N}_0^* \cup \mathcal{N}_1^* x_r \cup \cdots \cup \mathcal{N}_d^* x_r^d$. We claim that $\mathcal{N}^*$ is a strongly stable set of monomials.
Observe that it suffices to show that $\{x_1, \ldots, x_{r-1}\}\mathcal{N}_j^* \subset \mathcal{N}_{j-1}^*$. By using that $\mathcal{N}$ is a strongly stable set, we have that $\{x_1, \ldots, x_r\}\mathcal{N}_j \subset \mathcal{N}_{j-1}$. Then, by Lemma 3.3.10 and the induction hypothesis, we have that

$$|\{x_1, \ldots, x_{r-1}\}\mathcal{N}_j^*| = \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j^*) \leq \sum_{i=1}^{r-1} \gamma_{\leq i}(\mathcal{N}_j) = |\{x_1, \ldots, x_{r-1}\}\mathcal{N}_j| \leq |\mathcal{N}_{j-1}| = |\mathcal{N}_{j-1}^*|.$$

The fact that $|\{x_1, \ldots, x_{r-1}\}\mathcal{N}_j^*|$ and $|\mathcal{N}_{j-1}^*|$ are both lexsegments forces $|\{x_1, \ldots, x_{r-1}\}\mathcal{N}_j^*| \subset |\mathcal{N}_{j-1}^*|$, which implies that $\mathcal{N}^*$ is a strongly stable set of monomials.
Now, given a monomial $m = \prod_{i=1}^{r} x_i^{a_i}$, we set $\overline{m} = (x_{n-1}/x_n)^{a_n} m$. Observe that if $m_1 \leq m_2$ in the lexicographic order, then $\overline{m_1} \leq \overline{m_2}$.
Let $m_1 = \min \mathcal{L}$ and $m_2 = \min \mathcal{N}^*$. Since $\mathcal{N}^*$ is strongly stable, $\overline{m_2} \in \mathcal{N}_0^*$ and $\overline{m_2} \geq \min(\mathcal{N}_0^*)$. Further, $\min(\mathcal{N}_0^*) \geq m_2$, which implies that $\overline{\min(\mathcal{N}_0^*)} = \min(\mathcal{N}_0^*) \geq \overline{m_2}$. Hence, $\min(\mathcal{N}_0^*) = \overline{m_2}$ and similarly, $\min(\mathcal{L}_0^*) = \overline{m_1}$.
Since $|\mathcal{L}| \leq |\mathcal{N}| = |\mathcal{N}^*|$, we have that $m_1 \geq m_2$ and hence, $\overline{m_1} \geq \overline{m_2}$. As $\mathcal{L}_0$ and $\mathcal{N}_0^*$ are lexsegments, we get that $|\mathcal{L}_0| \leq |\mathcal{N}_0^*| = |\mathcal{N}_0|$, which completes the proof. □

We now complete the proof of Theorem 3.3.7.
Recall that we may assume that $I$ is strongly stable. Let $\mathcal{N}_j$ be the strongly stable set of monomials which spans the $\mathsf{k}$-vector space $I_j$. Since $|\mathcal{L}_j| = |\mathcal{N}_j|$, Bayer's theorem together with Lemma 3.3.10 implies that

$$|\mathrm{Shad}(\mathcal{L}_j)| = \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{L}_j) \leq \sum_{i=1}^{r} \gamma_{\leq i}(\mathcal{N}_j) = |\mathrm{Shad}(\mathcal{N}_j)|.$$

Since $I$ is an ideal, we have that $\mathrm{Shad}(\mathcal{N}_j) \subset \mathcal{N}_{j+1}$. Hence,

$$|\mathrm{Shad}(\mathcal{L}_j)| \leq |\mathrm{Shad}(\mathcal{N}_j)| \leq |\mathcal{N}_{j+1}| = |\mathcal{L}_{j+1}|.$$

Since $\mathrm{Shad}(\mathcal{L}_j)$ and $\mathcal{L}_{j+1}$ are both lexsegments, $|\mathrm{Shad}(\mathcal{L}_j)| \leq |\mathcal{L}_{j+1}|$ implies that $\mathrm{Shad}(\mathcal{L}_j) \subset \mathcal{L}_{j+1}$, as desired.

# Chapter 4

# The Auslander-Buchsbaum-Serre Theorem

In this chapter, we prove a result analogous to Hilbert's Syzygy Theorem (Theorem 2.4.7).

**Theorem 4.1.1.** *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a Noetherian local ring and* $\mu(\mathfrak{m}) = n$*. Then* $\operatorname{pdim}_R(\mathsf{k}) \geq n$ *and* $\beta_i^R(\mathsf{k}) \geq \binom{n}{i}$ *for all* $i \in \{0, \ldots, n\}$*. In particular, if* $R$ *is a regular local ring, then* $\operatorname{depth}(R) \geq \mu(\mathfrak{m})$*.*

*Proof.* Let $\{x_1, \ldots, x_n\}$ be a minimal generating set of $\mathfrak{m}$. Therefore, $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ for all $i$. We construct a minimal free resolution of $\mathsf{k}$ step by step.
Let $F_1 := \wedge^1 R^n$ denote $\oplus_{i=1}^n Re_i$. Define $\phi_1 : \wedge^1 R^n \to R$ as $\phi_1(e_i) = x_i$. Let

$$v_{ij} := x_i e_j - x_j e_i \in F_1, \text{ for all } 1 \leq i < j \leq n.$$

Note that each $v_{ij} \in \ker(\phi_1)$. We claim that the set $\{v_{ij} | 1 \leq i < j \leq n\}$ can be extended to a minimal generating set of $K_1 := \ker(\phi_1)$. Indeed, suppose there exists $\{a_{ij} \in R \mid 1 \leq i < j \leq n\}$ such that $\sum\limits_{1 \leq i < j \leq n} a_{ij} v_{ij} \in \mathfrak{m} K_1$. Since $F_1$ maps minimally onto $\mathfrak{m}$, $K_1 \subseteq \mathfrak{m} F_1$, and hence $\mathfrak{m} K_1 \subseteq \mathfrak{m}^2 F_1$. Suppose, for some $1 \leq i' < j' \leq n$ we have $a_{i'j'} \notin \mathfrak{m}$. Observe that the coefficient of $e_{j'}$ in $\sum\limits_{1 \leq i < j \leq n} a_{ij} v_{ij}$ is $\sum\limits_{i=1}^{j'-1} a_{ij'} x_i - \sum\limits_{i=j'+1}^{n} a_{j'i} x_i$, and hence $\sum\limits_{i=1}^{j'-1} a_{ij'} x_i - \sum\limits_{i=j'+1}^{n} a_{j'i} x_i \in \mathfrak{m}^2$. Since $a_{i'j'} \notin \mathfrak{m}$, it is a unit. Hence,

$$\{x_1, \ldots, x_{i'-1}, \sum_{i=1}^{j'-1} a_{ij'} x_i - \sum_{i=j'+1}^{n} a_{j'i} x_i, x_{i'+1}, \ldots, x_n\}$$

is also a minimal generating set of $\mathfrak{m}$ with one of the elements in $\mathfrak{m}^2$, which is a contradiction. Therefore, $a_{ij} \in \mathfrak{m}$ for all $1 \leq i < j \leq n$, which proves the claim.
Let $F_2 = R^{\beta_2^R(\mathsf{k})}$ be a free module mapping minimally onto $\ker(\phi_1)$. From what we have seen above, $\operatorname{rank}(F_2) = \beta_2^R(\mathsf{k}) \geq \binom{n}{2}$. Thus, we write $F_2 = \wedge^2 R^n \oplus G_2$.

Inductively assume that $\{v_{i_1\dots i_r} \mid 1 \leq i < j \leq n\}$ form a part of minimal generating set of $\ker(\phi_{r-1})$, where

$$v_{i_1\dots i_r} = \sum_{k=1}^{r}(-1)^{k-1}x_{i_k}e_{i_1}\wedge\cdots\wedge e_{i_{k-1}}\wedge e_{i_{k+1}}\wedge\cdots\wedge e_r \in F_{r-1} = \wedge^{r-1}R^n \oplus G_{r-1}.$$

Let $F_r$ denote a free module mapping minimally onto $\ker(\phi_{r-1})$. Then we have $\operatorname{rank}(F_r) = \beta_r^R(\mathsf{k}) \geq \binom{n}{r}$. We can decompose $F_r$ as $\wedge^r R^n \bigoplus G_r$, where $G_r$ is an $R$-free module of rank $\beta_r^R(\mathsf{k}) - \binom{n}{r}$. Let $\phi_r : F_r \to F_{r-1}$ be such that

$$\phi_r(e_{i_1}\wedge\cdots\wedge e_{i_r}) = v_{i_1\dots i_r}.$$

For all $1 \leq i_1 < \cdots < i_{r+1} \leq n$, let

$$v_{i_1\dots i_{r+1}} = \sum_{k=1}^{r+1}(-1)^{k-1}x_{i_k}e_{i_1}\wedge\cdots\wedge e_{i_{k-1}}\wedge e_{i_{k+1}}\wedge\cdots\wedge e_{i_{r+1}}.$$

Note that for all $1 \leq i_1 < \cdots < i_{r+1} \leq n$, $v_{i_1\dots i_{r+1}} \in \ker(\phi_r)$. Let $K_r$ denote $\ker(\phi_r)$. We claim that

$$\{v_{i_1\dots i_{r+1}} \mid 1 \leq i_1 < \cdots < i_{r+1} \leq n\}$$

forms a part of minimal generating set of $K_r$. To prove the claim, suppose that

$$\{a_{i_1\dots i_{r+1}} \in R \mid 1 \leq i_1 < \cdots < i_{r+1} \leq n\}$$

be such that $\sum a_{i_1\dots i_{r+1}}v_{i_1\dots i_{r+1}} \in \mathfrak{m}K_r$. Since $F_r$ maps minimally onto $\ker(\phi_{r-1})$, we must have $K_r \subseteq \mathfrak{m}F_r$ and hence, $\mathfrak{m}K_r \subseteq \mathfrak{m}^2F_r$. Suppose, for some $1 \leq i'_1 < \cdots < i'_{r+1} \leq n$, $a_{i'_1\dots i'_{r+1}} \notin \mathfrak{m}$. Observe that the coefficient of $e_{i'_2}\wedge\dots\wedge e_{i'_{r+1}}$ in $\sum a_{i_1\dots i_{r+1}}v_{i_1\dots i_{r+1}}$ is

$$\sum_{i_1=1}^{i'_2-1} a_{i_1 i'_2 i'_3\dots i'_{r+1}}x_{i_1} - \sum_{i_1=i'_2+1}^{i'_3-1} a_{i'_2 i_1 i'_3\dots i'_{r+1}}x_{i_1} + \cdots + (-1)^r \sum_{i_1=i'_{r+1}+1}^{n} a_{i'_2 i'_3\dots i'_{r+1} i_1}x_{i_1},$$

which must belong to $\mathfrak{m}^2$. Since $a_{i'_1 i'_2\dots i'_{r+1}} \notin \mathfrak{m}$, this contradicts the assumption that $\{x_1,\dots,x_n\}$ is a minimal generating set of $\mathfrak{m}$. Therefore, the set $\{v_{i_1\dots i_{r+1}} \mid 1 \leq i_1 < \cdots < i_{r+1} \leq n\}$ can be extended to a minimal generating set of $K_r$. Hence, $\operatorname{rank}(F_{r+1}) \geq \binom{n}{r+1}$.
If $R$ is regular local, by the Auslander-Buchsbaum formula, $\operatorname{depth}(R) = \operatorname{pdim}_R(\mathsf{k})$ and hence $\operatorname{depth}(R) \geq \mu(\mathfrak{m})$. □

Recall the following result.

**Lemma 4.1.2.** *Let $(R,\mathfrak{m},\mathsf{k})$ be a Noetherian local ring. Let $\{b_1,\dots,b_n\}$ be a minimal generating set of $\mathfrak{m}$, and $K_\bullet(b_1,\dots,b_n)$ be the corresponding Koszul complex. Then* $\operatorname{depth}(R) = \min\{j \mid H_{n-j}(K_\bullet(b_1,\dots,b_n)) \neq 0\}$.

**Corollary 4.1.3.** *Let $(R,\mathfrak{m},\mathsf{k})$ be a regular local ring. Then* $\operatorname{depth}(R) = \mu(\mathfrak{m})$.

*Proof.* Note that the above lemma implies that $\operatorname{depth}(R) \leq \mu(\mathfrak{m})$. Hence, we have $\operatorname{depth}(R) = \mu(\mathfrak{m})$. □

**Corollary 4.1.4.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a regular local ring. Then every minimal generating set of $\mathfrak{m}$ is a regular sequence.*

*Proof.* From the above lemma, since $\operatorname{depth}(R) = \mu(\mathfrak{m})$, we have that the Koszul complex corresponding to a minimal generating set of $\mathfrak{m}$ must be exact. □

**Theorem 4.1.5** (Auslander-Buchsbaum-Serre)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring with $\operatorname{depth}(R) = d$. The following statements are equivalent:*
*(i)* $\operatorname{pdim}_R(\mathsf{k}) < \infty$.
*(ii)* $\operatorname{pdim}_R(M) < \infty$ *for all finitely generated $R$-modules $M$.*
*(iii) $\mathfrak{m}$ is generated by a regular sequence.*
*(iv) $\mathfrak{m}$ is generated by $d$ elements.*
*(v)* $\operatorname{pdim}_R(\mathsf{k}) = d$.

*Proof.* (i) $\Longrightarrow$ (ii) Let $\operatorname{pdim}_R(\mathsf{k}) = n < \infty$. Then we have $\operatorname{Tor}_i^R(M, \mathsf{k}) = 0$ for all $i > n$ and for all finitely generated $R$-modules $M$. Therefore, $\operatorname{pdim}_R(M) < \infty$ for all finitely generated $R$-modules $M$.
(ii) $\Longrightarrow$ (i) is obvious since $k$ is a finitely generated $R$-module.
(i) $\Longrightarrow$ (iii) follows from Corollary 4.1.4.
(iii) $\Longrightarrow$ (i) If $\mathfrak{m}$ is generated by a regular sequence $x_1, \ldots, x_n$, then the Koszul complex on $x_1, \ldots, x_n$ gives a free resolution of $\mathsf{k}$ of finite length, which proves $\operatorname{pdim}_R(\mathsf{k}) = n < \infty$.
(i) $\Longrightarrow$ (iv) From Theorem 4.1.1 we know that $\operatorname{depth}(R) \geq \mu(\mathfrak{m})$. Hence, $d \geq \mu(\mathfrak{m})$. Thus $\mathfrak{m}$ is generated by $d$ elements.
(i) $\Longrightarrow$ (v) If $\operatorname{pdim}_R(\mathsf{k}) < \infty$, then by Auslander-Buchsbaum formula we get $\operatorname{pdim}_R(\mathsf{k}) = \operatorname{depth}(R) = d$.
(v) $\Longrightarrow$ (i) is obvious.
(iv) $\Longrightarrow$ (i) Let $\mathfrak{m}$ be generated by $d$ elements, that is, let $\mu(\mathfrak{m}) \leq \operatorname{depth}(R)$. From Lemma 4.1.2 we have $\operatorname{depth}(R) \leq \mu(\mathfrak{m})$. Therefore $\operatorname{depth}(R) = \mu(\mathfrak{m}) = d$. If $x_1, \ldots, x_d$ is a minimal generating set of $\mathfrak{m}$, then by Lemma 4.1.2 we have that the Koszul complex on $x_1, \ldots, x_d$ is exact, and hence is a free resolution of $\mathsf{k}$ of length $d$. This shows that $\operatorname{pdim}_R(\mathsf{k}) = n < \infty$. □

**Proposition 4.1.6.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a regular local ring such that* $\operatorname{depth}(R) = 2$. *Then $\mu(\mathfrak{m}) = 2$, and $\mathfrak{m}$ is generated by a regular sequence.*

*Proof.* By Auslander-Buchsbaum formula we have $\operatorname{depth}(R) = \operatorname{pdim}_R(\mathsf{k}) = 2$. We know that $\mu(\mathfrak{m}) \neq 1$, otherwise we would have $\operatorname{depth}(R) = 1$. Hence, $\mu(\mathfrak{m}) \geq 2$, and by Theorem 4.1.1 we have $\mu(\mathfrak{m}) = 2$. Suppose that $\mathfrak{m} = \langle x_1, x_2 \rangle$. We show that $x_1, x_2$ form a regular sequence by showing that the Koszul complex $K(x_1, x_2)$ is exact. Consider the Koszul complex on $x_1, x_2$ as follows:

$$0 \to R \xrightarrow{\phi_2} R^2 \xrightarrow{\phi_1} R \to R/\mathfrak{m} \to 0.$$

Since $\text{pdim}_R(\mathsf{k}) = 2$ and since $R^2$ maps minimally onto $\mathfrak{m}$, we see that $\ker(\phi_1)$ is free. By the Hilbert-Burch theorem, $\text{rank}(\ker(\phi_1)) = 1$. Let $\ker(\phi_1) = \langle (a_1, a_2) \rangle$, where $a_1, a_2 \in \mathfrak{m}$. Since $(-x_2, x_1) \in \ker(\phi_1)$, there exists $c \in R$ such that $(-x_2, x_1) = c(a_1, a_2)$. Since $x_1, x_2 \in \mathfrak{m} \setminus \mathfrak{m}^2$, we must have $c \notin \mathfrak{m}$, and hence $\ker(\phi_1) = \langle (-x_2, x_1) \rangle$. Therefore, the Koszul complex $K(x_1, x_2)$ is exact, and hence $x_1, x_2$ is regular. □

**Proposition 4.1.7.** *Let $R$ be a UFD, and let $I$ be an ideal of $R$ such that $\mu(R) = 2$. Then, $\beta_2^R(R/I) = 1$.*

*Proof.* Let $I = \{x_1, x_2\}$, and let $a_1, a_2 \in R \backslash \{0\}$ such that $a_1x_1 + a_2x_2 = 0$. Let $a = gcd(a_1, a_2)$, and $b_1 = a_1/a, b_2 = a_2/a$. Note that $b_1x_1 + b_2x_2 = 0$, and $b_1, b_2$ are coprime.
Suppose there exist $k_1, k_2 \in R$ such that $k_1x_1 + k_2x_2 = 0$.

$$k_1x_1 + k_2x_2 = 0 \implies b_1k_1x_1 + b_1k_2x_2 = 0 \implies -b_2k_1x_2 + b_1k_2x_2 = 0 \implies b_2k_1 = b_1k_2.$$

Since $b_1$ and $b_2$ are co-prime, $b_1|k_1$. Let $k_1 = kb_1$, and thus $k_2 = kb_2$. Hence, the kernel of the map from $R_2$ to $R$, which maps the basis elements of $R_2$ to a minimal generating set of $I$ must be a cyclic $R$-module. This implies that $\beta_2^R = 1$. □

# Chapter 5

# Existence of bounds on projective dimension and regularity

## 5.1 Burch's construction

To start with, I recall a couple of results which will be instrumental in the following proof.

**Lemma 5.1.1.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring. Let $I$ be an ideal whose associated primes are minimal over $I$. If $\mathfrak{P} \in \mathrm{Ass}(I)$, $I_P \cap R$ is the $P$-primary component in the minimal irredundant primary decomposition of $I$.*

**Proposition 5.1.2.** *(Macaulay's unmixedness theorem) In a Cohen-Macaulay ring $R$, the ideal $I$ generated by a regular sequence is unmixed, that is, all associated primes of $R/I$ have the same height as $I$.*

**Theorem 5.1.3** (Burch)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay ring and let $s$ be an integer such that $1 \leq s \leq \mathrm{depth}(R)$. Then there exists an ideal $I_s$ of $R$, generated by three or fewer elements of $R$, such that* $\mathrm{pdim}(R/I_s) = s$.

*Proof.* If $\mathrm{depth}(R) \leq 3$, let $I_s$ be the ideal generated by a regular sequence of length $s$.
Suppose $\mathrm{depth}(R) = d > 3$. We inductively construct a sequence $\{g_1, \ldots, g_{2(d-2)}\}$ of elements of $R$.
Let $g_1, g_2, g_3, g_4$ be a regular sequence in $R$. Consider the short exact sequence

$$0 \to (g_1, g_2) \cap (g_3, g_4) \to (g_1, g_2) \oplus (g_3, g_4) \to (g_1, g_2, g_3, g_4) \to 0.$$

Since $\mathrm{pdim}(R/(g_1, g_2)) = \mathrm{pdim}(R/(g_3, g_4)) = 2$ and $\mathrm{pdim}(R/(g_1, g_2, g_3, g_4)) = 4$, we have that $\mathrm{pdim}(R/(g_1, g_2) \cap (g_3, g_4)) = 3$ (follows directly by observing the induced long exact sequence of Tor modules).
Suppose we have chosen $g_1, g_2, \ldots, g_{2k}$ $(k < d-2)$ satisfying the conditions that
(i) $g_{2i-1}, g_{2i}, g_{2j-1}, g_{2j}$ is a regular sequence for all $i \leq j \leq k$.

(ii) $\operatorname{pdim}(R/(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k})) = k + 1$.
By the Auslander-Buchsbaum formula, $\operatorname{depth}(R/(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k})) = d - k - 1 \geq 2$. Let $g_{2k+1}$ be a nonzerodivisor and non-unit in $R/(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k})$. Then,

$$\operatorname{depth}\frac{R}{(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1})} \geq 1.$$

Hence, the ideal $(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1})$ has no $\mathfrak{m}$-primary component. Also, since $\operatorname{depth}(R) > 3$, for any $1 \leq i \leq k$, $\operatorname{depth}(R/(g_{2i-1}, g_{2i}, g_{2k+1}) > 0$ and $(g_{2i-1}, g_{2i}, g_{2k+1})$ has no $\mathfrak{m}$-primary component. Since $R$ is Noetherian, all ideals of $R$ have finitely many associated primes, and we can pick $g_{2k+2}$ to be a non-unit in no associated prime of $(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1})$ and in no associated prime of $(g_{2i-1}, g_{2i}, g_{2k+1})$ for any $i \leq k$. Then,

$$\operatorname{pdim}\frac{R}{(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) + (g_{2k+1}, g_{2k+2})} = k + 3,$$

and hence, by observing the induced long exact sequence of Tor modules, we have that

$$\operatorname{pdim}\frac{R}{(g_1, g_2) \cap \cdots \cap (g_{2k-1}, g_{2k}) \cap (g_{2k+1}, g_{2k+2})} = k + 2.$$

Thus, we can construct a sequence $\{g_1, \ldots, g_{2d-4}\}$ such that $\{g_1, \ldots, g_{2k}\}$ satisfies the above conditions for all $k \leq d - 2$.
Observe that $(g_{2i-1}, g_{2i}) : (g_j) = (g_{2i-1}, g_{2i})$ for all $i \leq 2d - 4$ and $j \neq 2i, 2i - 1$. Hence, for each associated prime $\mathfrak{P}$ of $(g_{2i-1}, g_{2i})$,

$$(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})_{\mathfrak{P}} = (g_{2i-1}, g_{2i})_{\mathfrak{P}}.$$

It follows that $\mathfrak{P}R_{\mathfrak{P}}$ is an associated prime of $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})_{\mathfrak{P}}$ in $R_{\mathfrak{P}}$ and thus, $\mathfrak{P}$ is an associated prime of $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ in $R$. However, $\prod_{j \leq 2s-4, j \notin \{2i-1, 2i\}} g_j$ is a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$ and a zerodivisor in $R/(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$. This implies that $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ has an associated prime which is not an associated prime of $(g_{2i-1}, g_{2i})$.
Note that $g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4}$ is a regular sequence. Indeed, if $g_1 g_3 \ldots g_{2d-5} h_1 = g_2 g_4 \ldots g_{2d-4} h_2$ for $h_1, h_2 \in R$, then $g_1 g_3 \ldots g_{2d-7} h_1 = g_{2d-4} h_3$, as $\{g_{2d-5}, g_{2d-4}\}$ is a regular sequence. Since $\{g_{2d-7}, g_{2d-4}\}$ is also a regular sequence, $g_1 g_3 \ldots g_{2d-9} h_1 = g_{2d-4} h_4$. Proceeding in this manner, we get $h_1 = g_{2d-4} f_1$ for some $f_1 \in R$ and similarly, $h_2 = g_{2d-5} f_2$ for some $f_2 \in R$. Hence, $g_1 g_3 \ldots g_{2d-7} f_1 = g_2 g_4 \ldots g_{2d-6} f_2$ and we are done by induction.
Hence, all associated primes of $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ are minimal over $(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4})$ and have height two. Also note that $g_j$ is not contained in any associated prime of $(g_{2i-1}, g_{2i})$ for any $i$, which forces that no associated prime of $(g_{2i-1}, g_{2i})$ is an associated prime of $(g_{2j-1}, g_{2j})$. Hence,

$$(g_1 g_3 \ldots g_{2d-5}, g_2 g_4 \ldots g_{2d-4}) = (g_1, g_2) \cap \cdots \cap (g_{2d-5}, g_{2d-4}) \cap J$$

where $J = \cap_{\mathfrak{P}\in\Lambda}((g_1g_3\dots g_{2d-5}, g_2g_4\dots g_{2d-4})_{\mathfrak{P}}\cap R)$, where $\Lambda = \text{Ass}((g_1g_3\dots g_{2d-5}, g_2g_4\dots g_{2d-4}))\setminus \cup_{1\leq i\leq d-2}\text{Ass}(g_{2i-1}, g_{2i})$.
Fix $i \leq d-2$. Observe that since every associated prime of $J$ has height 2 and is not an associated prime of $(g_{2i-1}, g_{2i})$, it must contain a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$. Since every primary component of $J$ contains a power of an associated prime of $J$, it also contains a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$. The product of the nonzerodivisors corresponding to every primary component of $J$ produces an element in $J$ which is a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$. Hence, $(g_{2i-1}, g_{2i}) : J = (g_{2i-1}, g_{2i})$ for $1 \leq i \leq d-2$.
By prime avoidance, there exists $x_d \in J$ such that $x_d$ is a nonzerodivisor in $R/(g_{2i-1}, g_{2i})$ for $1 \leq i \leq d-2$. Then,

$$(g_1g_3\dots g_{2d-5}, g_2g_4\dots g_{2d-4}) : x_d = (g_1, g_2) \cap \dots \cap (g_{2d-5}, g_{2d-4}).$$

Since the projective dimension of $(g_1g_3\dots g_{2d-5}, g_2g_4\dots g_{2d-4})$ is two, the short exact sequence

$$0 \to \frac{R}{I : x_d} \xrightarrow{x_d} \frac{R}{I} \to \frac{R}{(I, x_d)} \to 0$$

(where $I = (g_1g_3\dots g_{2d-5}, g_2g_4\dots g_{2d-4})$) gives us that

$$\text{pdim}\frac{R}{(g_1g_3\dots g_{2d-5}, g_2g_4\dots g_{2d-4}, x_d})) = s.$$

□

The above theorem tells us that we cannot hope to bound projective dimension as a function of the number of generators. This raises the question of whether we can achieve bounds as functions of the degrees of the generators.

## 5.2 Stillman's question and existence of bounds on regularity

**Stillman's question:** Let $\mathsf{k}$ be a field. Does there exist a bound, independent of $n$, on the projective dimension of an ideal in $S = \mathsf{k}[x_1, \dots, x_n]$ which is generated by $N$ forms of degrees $d_1, \dots, d_N$?
Stillman's question was answered in the affirmative by Ananyan and Hochster [1]. However, the bounds they produce are far from optimal. Optimal bounds have been given in the some cases, such as the following:

1. When $I$ is minimally generated by $N$ quadrics and $\text{ht}(I) = 2$, $\text{pdim}(S/I) \leq 2N-2$ [15].
2. When $I$ is minimally generated by four quadrics, $\text{pdim}(S/I) \leq 6$ [16].

3. When $I$ is minimally generated by three cubics, $\operatorname{pdim}(S/I) \leq 5$. [17].

**Definition 5.2.1.** *Let $R$ be a polynomial ring over a field and $M$ be a finitely generated graded $R$-module. The **Castelnuovo-Mumford regularity** of $M$ is defined as* $\operatorname{reg}_R(M) = \max\{j - i : \beta_{ij}(M) \neq 0\}$.

Note that $\operatorname{reg}(R/I) = \operatorname{reg}(I) - 1$ for an ideal $I \subset R$.
A question similar to Stillman's question can be asked on bounds on regularity.

**Question 1:** Let $\mathsf{k}$ be a field. Does there exist a bound, independent of $n$, on the regularity of an ideal in $S = \mathsf{k}[x_1, \ldots, x_n]$ which is generated by $N$ forms of degrees $d_1, \ldots, d_N$?
In fact, as outlined below, question 1 is equivalent to Stillman's question, if $\mathsf{k}$ is infinite. In fact, even if $\mathsf{k}$ is finite, we can consider the algebraic closure of $\mathsf{k}$ and arrive at the same conclusion.

Suppose Stillman's question has an affirmative answer, that is, there is a bound $B = B(N, d_1, \ldots, d_N)$ such that $\operatorname{pdim}(R/I) \leq B$ for any ideal $I \subset S = \mathsf{k}[x_1, \ldots, x_n]$ which is minimally generated by $N$ forms of degree $d_1, \ldots, d_N$. By the Auslander-Buchsbaum formula, $\operatorname{depth}(S/I) \geq n - B$. Let $\overline{f} = f_1, \ldots, f_{n-B}$ be a sequence of linear forms in $S$ which is regular in $S/I$. Such a sequence can be chosen because $\mathsf{k}$ is infinite. Since $S$ is a domain and $f_1, \ldots, f_{n-B}$ are linear forms, $f_1, \ldots, f_{n-B}$ is a regular sequence in $R$ as well. Hence, $\operatorname{reg}_S(S/I) = \operatorname{reg}_{S/(\overline{f})}(S/(I + (\overline{f})))$.
Now, $S/(\overline{f})$ is a polynomial ring in $B$ variables. There exists a bound on the regularity of $S/(I + (\overline{f}))$ in terms of $d(J) = \max\{d_1, \ldots d_N\}$ and the number of variables $B$ ([3], Theorem 3.8).

Conversely, assume that question 1 can be answered in the positive, that is, there exists a bound $B = B(N, d_1, \ldots, d_N)$ such that $\operatorname{reg}(I) \leq B$ for any ideal $I \subset S$ which is minimally generated by $N$ forms of degree $d_1, \ldots, d_N$. Consider $\operatorname{gin}_{\text{grevlex}}(I)$, the generic initial ideal of $I$ with respect to the graded reverse lexicographic order. By a theorem of Bayer and Stillman ([8], Corollaries 19.11 and 20.21),

$$\operatorname{pdim}(S/I) = \operatorname{pdim}(S/\operatorname{gin}_{\text{grevlex}}(I)), \operatorname{reg}(S/I) = \operatorname{reg}(S/\operatorname{gin}_{\text{grevlex}}(I)).$$

Moreover, the projective dimension of $S/\operatorname{gin}_{\text{grevlex}}(I)$ is the number of distinct variables appearing in all the monomials minimally generating $\operatorname{gin}_{\text{grevlex}}(I)$. Observe that for any ideal $J$ of $S$, we have that $d(J) \leq J$, where $d(J)$ denotes the maximal degree of a minimal generator of $M$. Hence,

$$\begin{aligned}
\operatorname{pdim}(S/I) &= \operatorname{pdim}(R/\operatorname{gin}_{\text{grevlex}}(I)) \\
&= \textit{number of distinct variables appearing in generators of } \operatorname{gin}_{\text{grevlex}}(I) \\
&\leq \textit{sum of degrees of generators of } \operatorname{gin}_{\text{grevlex}}(I) \\
&\leq (\textit{number of generators of } \operatorname{gin}_{\text{grevlex}}(I)) d(\operatorname{gin}_{\text{grevlex}}(I)) \qquad (5.1) \\
&\leq (\textit{number of generators of } \operatorname{gin}_{\text{grevlex}}(I)) \operatorname{reg}(\operatorname{gin}_{\text{grevlex}}(I)) \\
&= (\textit{number of generators of } \operatorname{gin}_{\text{grevlex}}(I)) \textit{reg}(I) \\
&\leq (\textit{number of generators of } \operatorname{gin}_{\text{grevlex}}(I)) B(N, d_1, \ldots, d_N).
\end{aligned}$$

The ideal $\text{gin}_{\text{grevlex}}(I)$ is generated by the initial terms of the elements of a Gröbner basis of $I$, after a generic change of co-ordinates. Note that a change of co-ordinates on $I$ does not change the number of generators of $I$ or the degrees of those generators. Hence, without loss of generality, we can assume that $I$ is in generic co-ordinates. To complete the proof, we need to bound the cardinality of a Gröbner basis of $I$ in terms of $N, d_1, \ldots, d_N$.
In a process similar to Buchberger's algorithm, we can attain a Gröbner basis by adjoining $S$-pairs of the form

$$S(f,g) = \frac{\text{LCM}(\text{in}(f), \text{in}(g))}{\text{in}(f)} f - \frac{\text{LCM}(\text{in}(f), \text{in}(g))}{\text{in}(g)} g.$$

Starting with $N$ generators, the maximum number of elements adjoined to the generating set on each iteration is a polynomial function in $N$. Further, $\deg(S(f,g)) \geq \max\{\deg(f), \deg(g)\}$ and this inequality is strict unless $\text{in}(f)$ divides $\text{in}(g)$ or vice-versa. On the other hand,

$$\begin{aligned} \deg(S(f,g)) &= \deg(\text{in}(S(f,g))) \\ &= d(\text{gin}_{\text{grevlex}}(I)) \\ &\leq \text{reg}(\text{gin}_{\text{grevlex}}(I)) \\ &= \text{reg}(I) \\ &\leq B. \end{aligned} \tag{5.2}$$

This limits the possible iterations in terms of $N, d_1, \ldots, d_N$. The proof is thus complete.

## 5.3 Regularity of modules over a Koszul algebra

**Lemma 5.3.1.** *Let $R$ and $M$ be as in Definition 5.2.1. Then,* $\text{reg}(M(-d)) = \text{reg}(M) + d$.

The proof follows immediately from the projective resolution of $M(-d)$.
Observe that one can think of regularity of a module as the height of its Betti table. The fact that the Betti table of $M(-d)$ is $d$ rows of zeroes above the Betti table of $M$ gives us another method of verifying the above lemma.
Note that $\text{reg}(M) = \max\{r : \exists i \textit{ such that } \text{Tor}_i^R(M, \mathsf{k})_{i+r} \neq 0\}$, where $\mathsf{k} = R/R_+$.

**Lemma 5.3.2.** *Let $R$ be a non-negatively graded ring and $0 \to A \to B \to C \to 0$ be a short exact sequence of $R$-modules. Then*
*(i)* $\text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\}$.
*(ii)* $\text{reg}(C) \leq \max\{\text{reg}(B), \text{reg}(A) - 1\}$.
*(iii)* $\text{reg}(A) \leq \max\{\text{reg}(B), \text{reg}(C) + 1\}$.

*Proof.* (i) Set $t = \text{reg}(B)$. Then, there exists $i$ such that $\text{Tor}_i^R(B, \mathsf{k})_{i+t} \neq 0$. The induced long exact sequence on Tor modules gives us the following exact sequence

$$\text{Tor}_{i+1}^R(C, \mathsf{k})_{i+t} \to \text{Tor}_i^R(A, \mathsf{k})_{i+t} \to \text{Tor}_i^R(B, \mathsf{k})_{i+t} \to \text{Tor}_i^R(C, \mathsf{k})_{i+t} \to \text{Tor}_{i-1}^R(A, \mathsf{k})_{i+t}.$$

Since $\operatorname{Tor}_i^R(B,\mathsf{k})_{i+t} \neq 0$, $\operatorname{Tor}_i^R(A,\mathsf{k})_{i+t} \neq 0$ or $\operatorname{Tor}_i^R(C,\mathsf{k})_{i+t} \neq 0$. Hence, $\operatorname{reg}(A) \geq t$ or $\operatorname{reg}(C) \geq t$.
(ii) Set $t = \operatorname{reg}(C)$. There exists $i$ such that $\operatorname{Tor}_i^R(C,\mathsf{k})_{i+t} \neq 0$. From the above exact sequence, $\operatorname{Tor}_{i-1}^R(A,k)_{i+t} \neq 0$ or $\operatorname{Tor}_i^R(B,\mathsf{k})_{i+t} \neq 0$. Hence, $t+1 \leq \operatorname{reg}(A)$ or $t \leq \operatorname{reg}(B)$.
(iii) Set $t = \operatorname{reg}(A)$. There exists $i$ such that $\operatorname{Tor}_i^R(A,\mathsf{k})_{i+t} \neq 0$. From the above exact sequence, $\operatorname{Tor}_{i+1}^R(C,\mathsf{k})_{i+t} \neq 0$ or $\operatorname{Tor}_i^R(B,\mathsf{k})_{i+t} \neq 0$. Hence, $t-1 \leq \operatorname{reg}(C)$ or $t \leq \operatorname{reg}(B)$. □

**Lemma 5.3.3.** *Let $R$ be a non-negatively graded ring and $M$ be a graded $R$-module. Suppose $x \in R_1$ is a nonzerodivisor on $M$. Then,* $\operatorname{reg}(M) = \operatorname{reg}(M/xM)$.

*Proof.* Consider the short exact sequence

$$0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0.$$

Let $t = \operatorname{reg}(M)$ and $s = \operatorname{reg}(M/xM)$. Then, by Lemma 5.3.1, $\operatorname{reg}(M(-1)) = t+1$. By (ii) of Lemma 5.3.2, $s \leq t$. Similarly, by (iii) of Lemma 5.3.2, $t+1 \leq s+1$. Hence, $s = t$. □

If $M$ is a graded $\mathsf{k}[x_1,\ldots,x_n]$-module of finite length, let $\max(M) = \max\{r : M_r \neq 0\}$.

**Proposition 5.3.4.** *Let $S = \mathsf{k}[x_1,\ldots,x_n]$, and let $M$ be a graded $S$-module of finite length. Then,*

$$\operatorname{reg}_S(M) = \max\{r : M_r \neq 0\}.$$

*Moreover, if $s = \operatorname{reg}_S(M)$,*

$$\operatorname{Tor}_n^S(M,\mathsf{k})_{n+s} \neq 0.$$

*Proof.* Consider the Koszul complex as a resolution of $\mathsf{k}$,

$$0 \to S(-n)^{b_n} \to S(-n+1)^{b_{n-1}} \to \cdots \to S(-1)^{b_1} \to S \to \mathsf{k} \to 0,$$

where $b_i = \binom{n}{i}$.
Let $s = \max(M)$. We have $\operatorname{Tor}_i^S(M,\mathsf{k}) \subset M(-i)^{b_i}$. Hence, $\max(\operatorname{Tor}_i^S(M,\mathsf{k})) \leq \max(M(-i)) = s+i$. Thus, $\operatorname{reg}(M) \leq s$. Further, note that

$$\operatorname{Tor}_n^S(M,\mathsf{k}) = \ker(M(-n) \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}} M(-n+1)^n).$$

Observe that $S_1 M_s \subset M_{s+1} = 0$. Hence, $0 \neq M(-n)_{s+n} \subset \operatorname{Tor}_n^S(M,\mathsf{k})$, which implies that $\operatorname{reg}_S(M) = s$ and $\operatorname{Tor}_n^S(M,\mathsf{k})_{n+s} \neq 0$. □

**Definition 5.3.5.** *A **Koszul algebra** $R$ is a graded $\mathsf{k}$-algebra over which the residue field $\mathsf{k}$ has a linear resolution, that is,* $\operatorname{reg}_R(\mathsf{k}) = 0$.

**Theorem 5.3.6.** *Let $R$ be a Koszul algebra, and let $Q = \mathsf{k}[R_1] = \mathsf{k}[x_1,\ldots,x_n]$. The regularity of any module $M$ over $R$ is finite. In fact,*

$$\operatorname{reg}_R(M) \leq \operatorname{reg}_Q(M).$$

*Proof.* We first prove the theorem in the case that $M$ has finite length. We proceed by induction on length of $M$.
In this case, $R/R_+ \cong \mathsf{k}$ injects into $M$ via multiplication by an element of $M$, say, $x$ of degree $d$. Let $N$ be the cokernel of this map. We have the short exact sequence

$$0 \to k(-d) \xrightarrow{x} M \to N \to 0.$$

By Lemma 5.3.2, $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(N)$ or $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(k(-d)) = d$ (by Lemma 5.3.1). If $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(N)$, we can apply the induction hypothesis to conclude that

$$\operatorname{reg}_R(M) \leq \operatorname{reg}_R(N) \leq \operatorname{reg}_Q(N) = \max(N) \leq \max(M) = \operatorname{reg}_Q(M).$$

On the other hand, if $\operatorname{reg}_R(M) \leq d$, then $\operatorname{reg}_R(M) \leq \max(M) = \operatorname{reg}_Q(M)$.
In the general case, we use Noetherian induction on the poset of submodules of M ordered by reverse inclusion. Hence, to prove that $\operatorname{reg}_R(M) \leq \operatorname{reg}_Q(M)$, it is sufficient to prove the following statement: Given a submodule $N \subset M$, if $\operatorname{reg}_R(M/N_1) \leq \operatorname{reg}_Q(M/N_1)$ for all $N_1 \supset N$, then $\operatorname{reg}_R(M/N) \leq \operatorname{reg}_Q(M/N)$. Without loss of generality, let $N = 0$.
If $R_+$ is not associated to $M$, then supposing as we may that $\mathsf{k}$ is infinite, there exists an element $x \in R_1$ such that $x$ is a nonzerodivisor on $M$. The result now follows from the induction hypothesis and 5.3.3.
If $M$ is not of finite length, but $R_+$ is associated to $M$, let $M'$ be a maximal submodule of finite length contained in $M$ and let $M'' = M/M'$ (note that $M' \neq 0$ because $\mathsf{k}$ injects into $M$). Then, $R_+$ is not associated to $M''$. Indeed, if $R_+$ was associated to $M''$, $\mathsf{k}$ would inject into $M''$ and hence $M''$ would contain a simple module, contradicting the maximality of $M'$. As in the proof of Proposition 5.3.4, $\operatorname{Tor}_n^S(M'', \mathsf{k}) = \operatorname{ann}''_M(S_+)(-n) = 0$. This implies that $\operatorname{Tor}_n^S(M, \mathsf{k}) = \operatorname{Tor}_n^S(M', \mathsf{k})$ and since $\operatorname{Tor}_n^S(M', \mathsf{k})_{n+\operatorname{reg}_S(M')} = 0$, we have $\operatorname{reg}_Q(M') \leq \operatorname{reg}_Q(M)$.
If $\operatorname{reg}_R(M'') \leq \operatorname{reg}_Q(M')$, by Lemma 5.3.2 and the finite length case treated above, we have that

$$\begin{aligned} \operatorname{reg}_R(M) &\leq \max\{\operatorname{reg}_R(M'), \operatorname{reg}_R(M'')\} \\ &\leq \operatorname{reg}_Q(M') \\ &\leq \operatorname{reg}_Q(M). \end{aligned} \tag{5.3}$$

If $\operatorname{reg}_R(M'') > \operatorname{reg}_Q(M')$, then by the induction hypothesis and the finite length case above,

$$\operatorname{reg}_R(M') \leq \operatorname{reg}_Q(M') < \operatorname{reg}_R(M'') \leq \operatorname{reg}_Q(M'').$$

Since $\operatorname{reg}_R(M') \leq \operatorname{reg}_R(M'')$, $\operatorname{reg}_R(M) \leq \operatorname{reg}_R(M'')$ by Lemma 5.3.2(i) and $\operatorname{reg}_R(M'') \leq \operatorname{reg}_R(M)$ by Lemma 5.3.2(ii). Hence, $\operatorname{reg}_R(M) = \operatorname{reg}_R(M'')$ and similarly, $\operatorname{reg}_Q(M) = \operatorname{reg}_Q(M'')$. By the induction hypothesis, we are done. □

# Chapter 6

# Pure Resolutions

## 6.1 Cohen-Macaulay modules with pure resolution

**Definition 6.1.1.** *Let $S = \mathsf{k}[x_1, \ldots, x_n]$ and let $M$ be a finitely generated non-negatively graded $R$-module. We say that $M$ has a **pure resolution** of type $(d_1, \ldots, d_p)$, where $0 < d_1 < \cdots < d_p$ is a strictly increasing sequence of non-negative integers, if $M$ has a minimal resolution of the form*

$$0 \to S(-d_p)^{\beta_p} \to S(-d_{p-1})^{\beta_{p-1}} \to \cdots \to S^{\beta_0} \to M \to 0.$$

**Theorem 6.1.2** (Herzog, Kühl)**.** *Let $S = \mathsf{k}[x_1, \ldots, x_n]$ and let $M$ be an $S$-module having a pure resolution of type $(d_1, \ldots, d_p)$ and Betti numbers $(\beta_0, \ldots, \beta_p)$, where $p$ is the projective dimension of $M$. Then the following conditions are equivalent:*
*(i) $M$ is Cohen-Macaulay.*
*(ii) $\beta_i = b_i\beta_0$ for $i = 1, \ldots, p$, where $b_i = (-1)^{i-1} \prod_{j \neq i} \frac{d_j}{d_j - d_i}$.*

*Proof.* Since the Hilbert series is additive on short exact sequences, and $H_{S(-d)}(z) = z^d/(1-z)^n$, the pure resolution

$$0 \to S(-d_p)^{\beta_p} \to S(-d_{p-1})^{\beta_{p-1}} \to \cdots \to S^{\beta_0} \to M \to 0$$

yields $H_M(z) = \sum_{i=0}^{p}(-1)^i\beta_i z^{d_i}/(1-z)^n$, where $d_0 = 0$. Recall that there exists a unique polynomial $R(z) \in \mathbb{Z}[z]$ such that $H_M(z) = R(z)/(1-z)^d$, where $d = \dim(M)$. Further, $d$ is the least integer $r$ such that $(1-z)^r H_M(z)$ is a polynomial. Let $m = n - d$, the codimension of $M$. By the Auslander-Buchsbaum formula, $m \leq p$ and $m = p$ iff $\operatorname{depth}(M) = d$, that is, $M$ is Cohen-Macaulay. We have that

$$(1-z)^m R(z) = \sum_{i=0}^{p}(-1)^i\beta_i z^{d_i}.$$

Suppose $M$ is Cohen-Macaulay. Then $(1-z)^p$ divides the right hand side of the above equation. Conversely, suppose that $(1-z)^p$ divides the right hand side. Then $(1-z)^{m-p+d}H_M(z)$ is a polynomial, which forces $m \geq p$. Hence, $m = p$ and $M$ is Cohen-Macaualay.

Thus, we need to prove that $(1-z)^p$ divides $\sum_{i=0}^{p}(-1)^i \beta_i z^{d_i}$ iff $\beta_i = b_i\beta_0$ for $i = 1,\ldots,p$, where $b_i = |\prod_{j\neq i}(d_j/d_{j-1})|$. Consider the polynomial $S(z) = 1 + \sum_{i=1}^{p} c_i z^{d_i}$ ($c_i \in \mathbb{Q}$). $(1-z)^p$ divides $S(z)$ iff $S(j)(1) = 0$ for $j = 0, 1, \ldots, p-1$ iff $c_1, \ldots, c_p$ satisfy the following system of linear equations

$$\begin{aligned} &\sum_{i=1}^{p} c_i = -1 \\ &\sum_{i=1}^{p} c_i d_i(d_i-1)\ldots(d_i-j+1) = 0 \end{aligned} \tag{6.1}$$

The matrix corresponding to this linear system is as follows

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ d_1 & d_2 & \ldots & d_p \\ d_1(d_1-1) & d_2(d_2-1) & \ldots & d_p(d_p-1) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ d_1(d_1-1)\ldots(d_1-p) & d_2(d_2-1)\ldots(d_2-p) & \ldots & d_p(d_p-1)\ldots(d_p-p) \end{pmatrix}.$$

On applying elementary row operations which do not affect the solution of the system, we obtain the Van der Monde matrix

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ d_1 & d_2 & \ldots & d_p \\ d_1^2 & d_2^2 & \ldots & d_p^2 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ d_1^{p-1} & d_2^{p-1} & \ldots & d_p^{p-1} \end{pmatrix}$$

with non-zero determinant $\prod_{i<j}(d_j - d_i)$. Let $A_i$ be the matrix obtained by replacing the $i^th$ column of $A$ by $(-1, 0, \ldots, 0)$. Then,

$$\det(A_i) = (-1)^i \prod_{j\neq i} d_j \prod_{k<j,\, j,k\neq i} (d_j - d_k).$$

By Cramer's rule,

$$c_i = (-1)^i \frac{\prod_{j\neq i} d_j}{\prod_{j<i}(d_i-d_j)\prod_{i<j}(d_j-d_i)} = -\prod_{j\neq i} \frac{d_j}{d_j-d_i}.$$

Hence, $(1-z)^p$ divides $\sum_{i=0}^{p}(-1)^i\beta_i z^{d_i}$ iff $(-1)^i\beta_i/\beta_0 = c_i$, that is, $\beta_i = (-1)^{i-1}\prod_{j\neq i}\frac{d_j}{d_j-d_i}\beta_0$. $\square$

## 6.2 Monomial ideals with linear resolution

Let $S = \mathsf{k}[x_1, \ldots, x_n]$ and $I \subset S$ be an equigenerated graded ideal, that is, a graded ideal whose generators $f_1, \ldots, f_k$ are all of the same degree. Then the Rees ring

$$R(I) = \oplus_{j \geq 0} I^j t^j = S[f_1 t, \ldots, f_k t] \subset S[t]$$

is naturally bigraded with $\deg(x_i) = (1, 0)$ for $i = 1, \ldots, n$ and $\deg(f_i t) = (0, 1)$ for $i = 1, \ldots, k$. Let $T = S[y_1, \ldots, y_k]$. We define a bigrading on $T$ by setting $\deg(x_i) = (1, 0)$ for $i = 1, \ldots, n$, and $\deg(y_j) = (0, 1)$ for $j = 1, \ldots, m$. Then there is a natural surjective homomorphism of bigraded $\mathsf{k}$-algebras $\phi : T \to R(I)$ with $\phi(x_i) = x_i$ for $i = 1, \ldots, n$ and $\phi(y_j) = f_j t$ for $j = 1, \ldots, k$.
Let

$$F_\bullet : 0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to R(I) \to 0$$

be the bigraded minimal free $T$-resolution of $R(I)$. Here $F_i = \oplus_j T(-a_{ij}, -b_{ij})$ for some $a_{ij}, b_{ij} \in \mathbb{Z}_{\geq 0}$, for $i = 0, \ldots, p$. Define the $x$-**regularity** of $I$ to be

$$\operatorname{reg}_x(R(I)) = \max_{i,j}\{a_{ij} - i\}.$$

Note that any homogeneous $f \in S$ has bidegree $(\deg(f), 0)$ as an element of $T$. Hence, given a bigraded $T$-module $M$, $M_{(*,n)}$ is a graded $S$-module for every $n$. It follows that for all $n$, the exact sequence $F_\bullet$ gives an exact sequence of graded $S$-modules

$$G_\bullet : 0 \to (F_p)_{(*,n)} \to \cdots \to (F_0)_{(*,n)} \to R(I)_{(*,n)} \to 0.$$

Note that considering $R(I)_{(*,n)}$ is isomorphic to $I^n$ as a $S$-module. However, as graded $S$-modules, $R(I)_{(*,n)}$ is isomorphic to $I^n(dn)$, because $(R(I)_{(*,n)})_a \cong I^n(a + dn)$ as $\mathsf{k}$-vector spaces (an element with bidegree $(a, n)$ in $R(I)$ is mapped to an element with bidegree $(a + dn, 0)$).
Also note that $(T(-a, -b))_{(*,n)}$ is isomorphic to the free $S$-module $\oplus_{|u|=n-b} S(-a) y^u$. It follows that $G_\bullet$ is a (possibly non-minimal) graded free $S$-resolution of $I^n(dn)$.
The following result is due to Römer ([21]).

**Theorem 6.2.1.** *With the notation introduced above,*

$$\operatorname{reg}(I^n) \leq nd + \operatorname{reg}_x(R(I)),$$

*for all $n \geq 0$. In particular, if $\operatorname{reg}_x(R(I)) = 0$, $I^n$ has a linear resolution for all $n \geq 0$.*

*Proof.* The resolution $G_\bullet$ above yields at once $\operatorname{reg}(I^n(dn)) \leq \operatorname{reg}_x(R(I))$, and hence, $\operatorname{reg}(I^n) \leq nd + \operatorname{reg}_x(R(I))$.
If $\operatorname{reg}_x(R(I)) = 0$, we have $\operatorname{reg}(I^n) \leq nd$. Since $I^n$ is generated in degree $nd$, $I^n$ must have a linear resolution. □

**Corollary 6.2.2.** *With notation as above, let $P = \ker(\phi)$. Then each power of $I$ has a linear resolution if for some monomial order $<$ on $T$, the ideal $P$ has a Gröbner basis $G$ whose elements are at most linear in the variables $x_1, \ldots, x_n$, that is, $\deg_x(f) \leq 1$ for all $f \in G$.*

*Proof.* The hypothesis implies that $\text{in}(P)$ is generated by monomials $u_1, \ldots, u_k$ with $\deg_x(u_i) \leq 1$. Let $C_\bullet$ be the Taylor resolution of $\text{in}(P)$. The module $C_i$ has basis $v_I$ with $I \subset \{1, \ldots, k\}$, $|I| = i$. Each basis element $e_\sigma$ has the multidegree $(a_I, b_I)$ where $x^{a_I} y^{b_I} = \text{LCM}\{u_j : j \in I\}$. It follows that $\deg_x(e_\sigma) \leq i$ for all $e_\sigma \in C_i$.
The shifts of $C_\bullet$ bound the shifts of a minimal multigraded resolution of $\text{in}(P)$, we conclude that $\text{reg}_x(T/\text{in}(P)) = 0$. By Corollary 3.1.19, $\text{reg}_x(T/P) \leq \text{reg}_x(T/\text{in}(P))$. Hence, $\text{reg}_x(T/P) = 0$ and the result follows from Theorem 6.2.1. □

Now, suppose $I$ is a squarefree monomial ideal generated in degree 2. We may associate to $I$ a graph $G$ whose vertices are numbered $1, \ldots, n$, and $\{i, j\}$ is an edge of $G$ iff $x_i x_j \in I$. The ideal $I$ is called the **edge ideal** of $G$ and denoted by $I(G)$. The assignment $G \to I(G)$ establishes a bijection between graphs and squarefree monomial ideals generated in degree 2.
The **complementary graph** $\overline{G}$ of $G$ is the graph on the same vertices, but whose edges are the non-edges of $G$. A graph $G$ is called **chordal** if each cycle of length greater than 3 has a chord.

**Theorem 6.2.3.** (Fröberg,[11]) *Given a graph $G$, $I(G)$ has a linear resolution iff $G$ is chordal.*

Let $\Delta$ be a simplicial complex, and denote by $\mathcal{F}(\Delta)$ the set of facets of $\Delta$. A facet $F \in \mathcal{F}(\Delta)$ is called a leaf if either $F$ is the only facet of $\Delta$, or there exists $G \in \mathcal{F}(\Delta)$, $G \neq F$ such that $H \cap F \subset G \cap F$ for each $H \in \mathcal{F}(\Delta)$ with $H \neq F$. A vertex $i$ of $\Delta$ is called a **free vertex** if $i$ belongs to precisely one facet.
A simplicial complex $\Delta$ is called a **quasi-tree** if there exists a labelling $F_1, \ldots, F_m$ of the facets such that for all $i$, the facet $F_i$ is a leaf of the subcomplex $\langle F_1, \ldots, F_i \rangle$. We call such a labeling a **leaf order**.

**Theorem 6.2.4.** (Dirac) *A graph $G$ is chordal iff $G$ is the 1-skeleton of a quasi-tree.*

**Proposition 6.2.5.** *Let $I \subset S$ be a squarefree monomial ideal with 2-linear resolution. Then after suitable renumbering of the variables, we have: if $x_i x_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either $x_i x_k$ or $x_j x_k$ belongs to $I$.*

*Proof.* Let $G$ be the graph such that $I(G) = I$. By Theorems 6.2.3 and 6.2.4, $\overline{G}$ is the 1-skeleton of a quasi-tree $\Delta$. Let $F_1, \ldots, F_m$ be a leaf order of $\Delta$. Let $i_1$ be the number of free vertices of the leaf $F_m$. We label the free vertices of $F_m$ by $n, n-1, \ldots, n-i_1+1$, in any order. Next $F_{m-1}$ is a leaf of $\langle F_1, \ldots, F_{m-1} \rangle$. Label the $i_2$ free vertices of $F_{m-1}$ by $n-i_1, \ldots, n-(i_1+i_2)+1$, in any order. Proceeding in this manner, we label all the vertices of $\Delta$, that is, those of $G$, and choose the numbering of the variables of $S$ according to this labeling.
Suppose there exist $i, j$ such that $x_i x_j \in I$ and $k > i, j$ such that $x_i x_k \notin I$ and $x_j x_k \notin I$. Let $r$ be the smallest number such that $\Gamma = \langle F_1, \ldots, F_r \rangle$ contains the vertices $1, \ldots, k$. Then, by the numbering of the variables, $k \in F_r$ is a free vertex in $\Gamma$.
Since $x_i x_k \notin I$, $\{i, k\}$ is an edge in $\Delta$. Suppose $\{i, k\}$ is not an edge in $\Gamma$. Let $p$ be the smallest number such that $F_p$ contains the edge $\{i, k\}$. Since $F_p$ is a leaf in $\langle F_1, \ldots, F_p \rangle$, there exists $q \in \{1, \ldots, p-1\}$ such that $i \in F_q$ and $k \in F_q$ (since $i$ and $k$ belong in the intersection of $F_p$ with other facets). This contradicts the choice of $p$. Hence, $\{i, k\}$ and similarly, $\{j, k\}$, are edges in $\Gamma$.

Since $F_r$ is the only facet containing the vertex $k$, $i$ and $j$ must be vertices of $F_r$ as well. However, this implies that $\{i, j\}$ is an edge of $F_r$, and hence of $\Delta$. This contradicts the assumption that $x_i x_j \in I$. □

We now consider a monomial ideal $I$ generated in degree 2 which is not necessarily squarefree. Let $J \subset I$ be the ideal generated by all squarefree monomials in $I$. Then $I = \langle x_{i_1}^2, \ldots, x_{i_k}^2, J\rangle$ for distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

**Lemma 6.2.6.** *If $I$ has a linear resolution, so does $J$.*

*Proof.* Polarizing (section 3.2) the ideal $I = \langle x_{i_1}^2, \ldots, x_{i_k}^2, J\rangle$ yields the ideal $I^* = \langle x_{i_1}, \ldots, x_{i_k}, y_1, \ldots, y_k, J\rangle$ in $\mathsf{k}[x_1, \ldots, x_n, y_1, \ldots, y_k]$. We consider $I^*$ as the edge ideal of the graph $G^*$ with vertices $-k, \ldots, -1, 1, \ldots, n$, where the vertices $-i$ correspond to the variables $y_i$ and as usual, the vertices $i$ correspond to the variables $x_i$. Let $G$ be the restriction of $G^*$ to the vertices $1, \ldots, n$. Then, $J$ is the edge ideal of $G$.
By Corollary 3.2.5, if $I$ has a linear resolution, so does $I^*$. Hence, by Theorem 6.2.3, $G^*$ is chordal. Thus, $G$ is chordal and by Theorem 6.2.3, $J$ has a linear resolution. □

In the situation of Lemma 6.2.6, let $J = I(G)$, and let $\Delta$ be the quasi-tree whose 1-skeleton is $\overline{G}$ (Theorem 6.2.4).

**Lemma 6.2.7.** *If $I = \langle x_{i_1}^2, \ldots, x_{i_k}^2, J\rangle$ has a linear resolution, then $i_j$ is a free vertex of $\Delta$ for $j = 1, \ldots, k$, and no two of these vertices belong to the same facet.*

*Proof.* The hypothesis implies that $\overline{G^*}$ is chordal.
Suppose $i_j$ is not a free vertex of $\Delta$ for some $j$. Then there exist edges $\{i_j, r\}$ and $\{i_j, s\}$ in $\overline{G}$ such that $\{r, s\}$ is not an edge in $\overline{G}$. Then $\{i_j, r\}$ and $\{i_j, s\}$ are also edges in $\overline{G^*}$ and $\{r, s\}$ is not an edge in $\overline{G^*}$. Since $x_{i_j} y_j \in I^*$, $\{i_j, -j\}$ is not an edge in $G^*$, and since $x_r y_j$ and $x_s y_j$ do not belong to $I^*$, it follows that $\{-j, r\}$ and $\{-j, s\}$ are edges in $\overline{G^*}$. Thus, $\{i_j, r\}$, $\{r, -j\}$, $\{-j, s\}$ and $\{s, i_j\}$ form a cycle in $\overline{G^*}$ of length 4 without any chords, a contradiction.
Suppose $i_j$ and $i_k$ are free vertices belonging to the same facet of $\Delta$. Then, $\{i_j, i_l\}$, $\{i_l, -j\}$, $\{-j, -l\}$ and $\{-l, i_j\}$ is a cycle in $\overline{G^*}$ without any chords. □

**Corollary 6.2.8.** *Suppose $I = \langle x_{i_1}^2, \ldots, x_{i_k}^2, J\rangle$ has a linear resolution and $x_i^2 \in I$. Then with the numbering of the variables as given by Proposition 6.2.5 (applied on $J$), the following holds: for all $j > i$ for which there exists $k$ such that $x_k x_j \in I$, we have $x_i x_j \in I$ or $x_i x_k \in I$.*

*Proof.* Suppose $x_i^2 \in I$ and there exists $j > i$ for which there exists $k$ such that $x_k x_j \in I$, but $x_i x_j$ and $x_i x_k$ do not belong to $I$. Then $i \neq k$.
If $k \neq j$, then $\{k, j\}$ is not an edge of $\Delta$, but $\{i, j\}$ and $\{i, k\}$ are. Hence, $i$ is not a free vertex of $\Delta$, which contradicts Lemma 6.2.7.
If $k = j$, then $x_j^2 \in I$ and $j$ is a free vertex in $\Delta$. However, since $\{i, j\}$ is an edge in $\Delta$, $i$ and $j$ must belong to the same facet, which contradicts Lemma 6.2.7. □

**Theorem 6.2.9.** *Let $I \subset S$ be a monomial ideal generated in degree 2 and suppose that $I$ possesses the following properties (*) and (**):*
*(*) if $x_ix_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either $x_ix_k$ or $x_jx_k$ belongs to $I$;*
*(**) if $x_i^2 \in I$, then for all $j > i$ for which there exists $k$ such that $x_kx_j \in I$, we have $x_ix_j \in I$ or $x_ix_k \in I$.*
*Let $R(I) = T/P$ be the Rees ring of $I$. Then there exists a lexicographic order $<_{\text{lex}}$ on $T$ such that the reduced Gröbner basis $G$ of the defining ideal $P$ with respect to $<_{\text{lex}}$ consists of binomials $f \in T$ with $\deg_x(f) \leq 1$.*

*Proof.* Let $\Omega$ denote the graph with vertices $1, \ldots, n+1$ whose edge set $E(\Omega)$ consists of those edges (and loops) $\{i, j\}$, $1 \leq i \leq j \leq n$, with $x_ix_j \in I$ together with the edges $\{1, n+1\}, \{2, n+1\}, \ldots, \{n, n+1\}$. Let $\mathsf{k}[\Omega]$ denote the affine semigroup ring generated by those quadratic monomials $x_ix_j$, $1 \leq i \leq j \leq n+1$, with $\{i, j\} \in E(\Omega)$. Let $T = \mathsf{k}[x_1, \ldots, x_n, \{y_{\{i,j\}}\}_{1\leq i\leq j\leq n, \{i,j\}\in E(\Omega)}]$ be the polynomial ring and define the surjective homomorphism $\pi : T \to \mathsf{k}[\Omega]$ by setting $\pi(x_i) = x_ix_{n+1}$ and $\pi(y_{\{i,j\}}) = x_ix_j$. The **toric ideal** of $\mathsf{k}[\Omega]$ is the kernel of $\pi$. Note that the Rees ring $R(I)$ is isomorphic to $\mathsf{k}[\Omega]$ and we can identify the defining ideal $P$ of the Rees ring with the toric ideal of $K[\Omega]$.
Introduce the lexicographic order $<_{\text{lex}}$ on $T$ induced by the ordering of the variables as follows:
(i) $y_{i,j} > y_{p,q}$ if (a) $\min\{i, j\} < \min\{p, q\}$ or (b) $\min\{i, j\} = \min\{p, q\}$ and $\max\{i, j\} < \max\{p, q\}$.
(ii) $y_{\{i,j\}} > x_1 > x_2 > \cdots > x_n$ for all $y_{\{i,j\}}$.
The Graver basis of an ideal is defined in [22] (Ch.4). It is proved in [18] that the Graver basis of a toric ideal $P$ coincides with the set of all binomials $f_\tau$ (notation explained below), where $\tau$ is a primitive even closed walk in $\Omega$. Further, the universal Gröbner basis (defined earlier) is contained in the Graver basis ([22], Proposition 4.11). A minimal Gröbner basis $G$ of $P$ with respect to $<_{\text{lex}}$ can be obtained as a subset of the universal Gröbner basis of $P$. It follows that every element of $G$ is of the form $f_\tau$, where $\tau$ is a primitive even closed walk in $\Omega$.
Let $f$ be a binomial belonging to $G$ and

$$\Gamma = (\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{2m}, w_1\})$$

be the primitive even closed walk associated to $f$. This means that setting $y_{i,n+1} = x_i$ and $w_{2m+1} = w_1$,

$$f = f_\Gamma = \prod_{k=1}^{m} y_{w_{2k-1},w_{2k}} - \prod_{k=1}^{m} y_{w_{2k},w_{2k+1}}.$$

We need to prove that $\deg_x(f_\tau) \leq 1$, that is, among the vertices $w_1, \ldots, w_{2m}$, the vertex $n+1$ appears at most once. Let $y_{w_1,w_2}$ be the biggest variable appearing in $f$ with respect to $<_{\text{lex}}$, with $w_1 \leq w_2$. Note that $\text{in}_{\text{lex}}(f_\Gamma) = \prod_{k=1}^{m} y_{\{w_{2k-1},w_{2k}\}}$. We denote this as in$(f_\Gamma$.
Note that $w_1 \neq n+1$ because $w_1 = n+1$ forces $w_2 = n+1$ and $y_{\{n+1,n+1\}} \notin T$. If $w_2 = \{n+1\}$, $y_{w_1,w_2}$ being the biggest variable in $f$ implies that $\Gamma$ must be $(\{w_1, w_2\}, \{w_2, w_1\})$, in which case $f_\Gamma = 0$. Thus, suppose that $w_2 < n+1$. Let $k_1$ be the smallest integer such that $w_{k_1} = n+1$.
**Case 1:** Suppose $k_1$ is even. Since $\{n+1, w_1\} \in E(\Omega)$, the closed walk

$$\Gamma' = (\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{k_1-1}, w_{k_1}\}, \{w_{k_1}, w_1\})$$

is an even closed walk in $\Omega$ with $\deg_x(f_{\Gamma'}) = 1$. Since $f_{\Gamma'} \in I$, $\text{in}(g)$ must divide $\text{in}(f_{\Gamma'}) = y_{\{w_1,w_2\}}y_{\{w_3,w_4\}} \cdots y_{\{w_{k_1-1},w_{k_1}\}}$, which divides $\text{in}(f_\Gamma)$, for some $g \in G$. Since $G$ is a minimal Gröbner basis, we have $\text{in}(g) = \text{in}(f_\Gamma)$ and hence, $\text{in}(f_{\Gamma'}) = \text{in}(f_\Gamma)$. Thus, $k_1 = 2m$ and the vertex $n+1$ appears only once in $\Gamma$.
**Case 2:** Suppose $k_1$ is odd. Suppose there exists $k_2 > k_1$ such that $w_{k_2} = n+1$. We can further assume that $w_i \neq n+1$ for $k_1 < i < k_2$.
*Case 2a:* Suppose $k_2$ is odd. Then consider the subwalk of $\Gamma$,

$$\Gamma'' = (\{w_1, \ldots, w_2\}, \ldots, \{w_{k_1-1}, w_{k_1}\}, \{w_{k_2}, w_{k_2+1}\}, \ldots, \{w_{2m}, w_1\}).$$

$\Gamma'$ is a closed even subwalk in $\Omega$, which contradicts that $\Gamma$ is a primitive even closed walk in $\Omega$.
*Case2b:* Suppose $k_2$ is even. Let $C$ be the odd closed walk

$$C = (\{w_{k_1}, w_{k_1+1}\}, \{w_{k_1+1}, w_{k_1+2}\}, \ldots, \{w_{k_2-1}, w_{k_2}\})$$

in $\Omega$. Since both $w_1$ and $w_2$ are not equal to $n+1$, $\{w_2, w_{k_1}\}$ and $\{w_1, w_{k_1}\}$ are edges in $\Omega$. Consider the even closed walk

$$\Gamma''' = (\{w_1, w_2\}, \{w_2, w_{k_1}\}, C, \{w_{k_2}, w_1\})$$

in $\Omega$. The initial monomial $\text{in}(f_{\Gamma'''})$ divides $\text{in}(f_\Gamma)$ and hence, $\text{in}(f_{\Gamma'''}) = \text{in}(f_\Gamma)$. The monomial $\text{in}(f_{\Gamma'''})$ has degree $\frac{k_2-k_1+1}{2} + 1$ and $\text{in}(f_\Gamma)$ has degree $m$. Equating the degrees, we get $k_2 - k_1 = 2m-3$, which forces $k_2 = 2m$ and $k_1 = 3$ because $3 \leq k_1 < k_2 \leq 2m$. Hence, $\Gamma''' = \Gamma$. We also have that $w_3 = w_{2m} = n+1$.
We claim that none of the vertices of $C$ coincides with $w_1$ or $w_2$. Given a vertex $w_i$, consider the two paths

$$C_1 = (\{w_3, w_4\}, \ldots, \{w_{i-1}, w_i\}),$$
$$C_2 = (\{w_{2m}, w_{2m-1}\}, \ldots, \{w_{i+1}, w_i\}).$$

Since $C$ is odd, one of $C_1$ and $C_2$ must be odd and the other must be even. Suppose $C_1$ is odd and $C_2$ is even. If $w_i = w_1$, then

$$(\{w_1, w_2\}, \{w_2, w_{2m}\}, C_2)$$

is an even closed walk in $\Omega$ and contradicts the assumption that $\Gamma$ is a primitive even closed walk. If $w_i = w_2$, then the walk

$$(\{w_2, w_3\}, C_1)$$

gives us a contradiction. A similar argument works if $C_1$ is even and $C_2$ is odd. Hence, $w_i \neq w_1$ and $w_i \neq w_2$ for $3 \leq i \leq 2m$.
*Case 2b(i):* Suppose there exists $p \geq 0$ with $3 + (p+2) < 2m$ such that $w_{3+(p+1)} \neq w_{3+(p+2)}$ (this is equivalent to supposing that the cycle $C$ contains at least 3 distinct vertices). Let $W_1$, $W_2$, $W_3$ and $W_4$ be the walks

$$W_1 = (\{w_3, w_4\}, \ldots, \{w_{4+p}, w_{5+p}\}),$$
$$W_2 = (\{w_{2m}, w_{2m-1}\}, \ldots, \{w_{6+p}, w_{5+p}\}),$$

$$W_3 = W_1 - \{w_{4+p}, w_{5+p}\},$$

$$W_4 = W_2 + \{w_{5+p}, w_{4+p}\}$$

in $\Omega$. Note that since $C$ is odd, one of $W_1$ and $W_2$ must be odd and the other must be even. Assume that $W_1$ is odd and $W_2$ is even. Then, $W_3$ is even and $W_4$ is odd. A similar argument works if $W_1$ is even and $W_2$ is odd.
Suppose $\{w_2, w_{4+p}\} \in E(\Omega)$ or $\{w_2, w_{5+p}\} \in E(\Omega)$. Then we can construct an even closed walk $\Gamma_1$ in $\Omega$ such that $\text{in}(f_{\Gamma_1})$ divides $\text{in}(f_\Gamma)$ and $\deg_x(f_{\Gamma_1}) = 1$. This walk is constructed as follows: suppose first that $\{w_2, w_{4+p}\} \in E(\Omega)$. Then,

$$\Gamma_1 = (\{w_2, w_1\}, \{w_1, w_{2m}\}, W_4, \{w_{4+p}, w_2\}).$$

If $\{w_2, w_{5+p}\} \in E(\Omega)$, then

$$\Gamma_1 = (\{w_2, w_1\}, \{w_1, w_3\}, W_1, \{w_{5+p}, w_2\}).$$

Note that $\text{in}(f_{\Gamma_1})$ divides $\text{in}(f_\Gamma)$ in both cases (even if $w_1 = w_2$, we have that $w_1 \leq w_{5+p}$ because of the minimality of $y_{w_1,w_2}$ and further, $w_1 < w_{5+p}$ because $3 \leq 5+p \leq 2m$). Hence, we must have $\text{in}(f_{\Gamma_1}) = \text{in}(f_\Gamma)$ and on comparing degrees, we get $\frac{p+5}{2} = m$. This leads to a contradiction because $p + 2 < 2m - 3$.
Suppose $\{w_2, w_{4+p}\} \notin E(\Omega)$ and $\{w_2, w_{5+p}\} \notin E(\Omega)$.
If $w_1 \neq w_2$, by (*), $w_2 < w_{4+p}$ or $w_2 < w_{5+p}$ (we already know that $w_{4+p}$ and $w_{5+p}$ are not equal to $w_2$). If $w_2 < w_{4+p}$, then since $w_1 < w_2$, we have $\{w_1, w_{4+p}\} \in E(\Omega)$. Then we can consider the even closed walk

$$\Gamma_2 = (\{w_1, w_2\}, \{, w_2, w_{2m}\}, W_4, \{w_{4+p}, w_1\})$$

in $\Omega$. Proceeding as above (and using that $w_2 < w_{4+p}$), we get $p = 2m - 5$, a contradiction. If $w_2 < w_{5+p}$, we have $\{w_1, w_{4+p}\} \in E(\Omega)$. In that case, consider the even closed walk

$$\Gamma_3 = (\{w_1, w_2\}, \{, w_2, w_{2m}\}, W_1, \{w_{5+p}, w_1\})$$

in $\Omega$ and proceed as above.
If $w_1 = w_2$, since $w_1 < w_{4+p+}$, by (**), $\{w_1, w_{4+p}\} \in E(\Omega)$ or $\{w_1, w_{5+p}\} \in E(\Omega)$. Then we can construct the walk $\Gamma_2$ or $\Gamma_3$ and proceed similarly.
*Case 2b(ii)*: The only case remaining is when $C = (\{n+1, j\}, \{j, j\}, \{j, n+1\})$. The three possibilities are $w_1 < w_2 < j$, $w_1 < j < w_2$ and $w_1 = w_2 < j$. On applying (*), (**) and (**) respectively, we get $\{w_1, j\} \in E(\Omega)$ or $\{w_2, j\} \in E(\Omega)$ in each case. If $\{w_1, j\} \in E(\Omega)$, consider the walk $\Gamma_4 = (\{w_1, w_2\}, \{w_2, n+1\}, \{n+1, j\}, \{j, w_1\})$. As before, we get $\text{in}_{\Gamma_4} = \text{in}_\Gamma$, which is not possible on comparing degrees. A similar argument works in $\{w_2, j\} \in E(\Omega)$. □

**Corollary 6.2.10.** *Let $I$ be a monomial ideal in $S$ generated in degree 2. Then, $I$ has a linear resolution iff each power of $I$ has a linear resolution.*

*Proof.* Follows immediately from Theorem 6.2.9 and Corollary 6.2.2. □

## 6.3 Associated graded modules with pure resolution

Let $A = \mathsf{k}[[x_1, \dots, x_n]]$ and $\mathfrak{m} = (x_1, \dots, x_n)$. Let $G_{\mathfrak{m}}(A)$ be the associated graded ring of $A$ with respect to $\mathfrak{m}$, that is,

$$G_{\mathfrak{m}}(A) = \bigoplus_{d \geq 0} \frac{\mathfrak{m}^d}{\mathfrak{m}^{d+1}}.$$

It is well known that $G_{\mathfrak{m}}(A) \cong \mathsf{k}[\mathsf{x}_1, \dots, \mathsf{x}_n] = S$. Let $M$ be a finitely generated $A$-module and $G_{\mathfrak{m}}(M) = \bigoplus_{d \geq 0} \mathfrak{m}^d M / \mathfrak{m}^{d+1} M$ be the associated graded module of $M$ with respect to $\mathfrak{m}$.
Note that $G_{\mathfrak{m}}(M)$ is a $S$-module. The goal of this section is to figure out when $G_{\mathfrak{m}}(M)$ has a free $S$-resolution.
Given any element $v \in M$, $v \in \mathfrak{m}^d M \setminus \mathfrak{m}^{d+1} M$ for some $d \geq 0$. Hence, there is we have an element in $G_{\mathfrak{m}}(M)$ which naturally corresponds to $v$, and we denote this element by $v* \in \mathfrak{m}^d M / \mathfrak{m}^{d+1} M$.
Let $\phi : A^l \to A^k$ be a non-zero $A$-linear map. There exists $s \geq 0$ such that $\mathrm{Im}(\phi) \subset \mathfrak{m}^s A^k$ and $\mathrm{Im}(\phi) \not\subset \mathfrak{m}^{s+1} A^k$. Let $\phi = (a_{ij})$ where $a_{ij} \in A$. By assumption, $a_{ij} \in \mathfrak{m}^s$. Consider $\phi^* : S^l \to S^k$, $\phi^* = (a^*_{ij})$. It follows that

$$\phi^* = \sum_{j \geq s} \phi^*_j,$$

where $\phi^*_j$ is a matrix with homogeneous entries of degree $j$. Set $\mathrm{in}(\phi) = \phi^*_s$. We call $\mathrm{in}(\phi)$ the **initial form** of $\phi$. Set $v(\phi) = s$, the **order** of $\phi$.
Let

$$\mathbb{F} : 0 \to F_p \xrightarrow{\phi_p}\to F_{p-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

be a minimal $A$-resolution of $M$. Let $c_i = v(\phi_i)$ and $d_i = \sum_{j=1}^{i} c_i$ for $i = 1, \dots, p$. Let $\beta_i = \beta_i(M)$. Since $\phi_i \circ \phi_{i+1} = 0$ and $\mathrm{in}(\phi_i)$ contains only the lowest degree terms of $\phi_i$, it follows that $\mathrm{in}(\phi_i) \circ \mathrm{in}(\phi_{i+1}) = 0$. Hence, we have a complex

$$\mathrm{in}(\mathbb{F}) : 0 \to S(-d_p)^{\beta_p} \xrightarrow{\mathrm{in}(\phi_p)} S(-d_{p-1})^{\beta_{p-1}} \to \cdots \to S(-d_1)^{\beta_1} \xrightarrow{\mathrm{in}(\phi_1)} S^{\beta_0} \to 0.$$

Note that if $M$ is minimally generated by $\{v_1, \dots, v_k\}$ as an $A$-module, then by Nakayama lemma, $v_i \notin \mathfrak{m}M$. The associated graded module $G_{\mathfrak{m}}(M)$ is minimally generated by $\{v_1^*, v_2^*, \dots, v_k^*\} \subset M/\mathfrak{m}M$ as a $S$-module. In this case, $\beta_0 = k$ and suppose that the map $\phi_0 : F_0 \to M$ was defined as $\phi_0(w_i) = v_i$.
We thus have a natural $S$-linear map $\epsilon : S^{\beta_0} \to G_{\mathfrak{m}}(M)$ defined as $\epsilon(w_i) = v_i^*$.

**Lemma 6.3.1.** *With notation as above, $\epsilon$ is surjective and $\epsilon \circ \mathrm{in}(\phi_1) = 0$.*

*Proof.* The map $\epsilon$ is surjective as $\{v_1^*, v_2^*, \dots, v_k^*\}$ is a generating set of $M$.
Let $\phi_1 = (a_{ij})$ and $\mathrm{in}(\phi_1) = (b_{ij})$. Since $\phi_0 \circ \phi_1 = 0$, $\sum_{i=1}^k a_{ij} v_i = 0$ for all $j$. We need to prove that $\sum_{i=1}^k b_{ij} v_i^* = 0$.
Suppose $v(\phi_1) = d$. Then, note that the component of $\sum_{i=1}^k a_{ij} v_i = 0$ in $\mathfrak{m}^d M \setminus \mathfrak{m}^{d+1} M$ is precisely the sum $\sum_{i=1}^k b_{ij} v_i^*$. Hence, $\sum_{i=1}^k b_{ij} v_i^* = 0$. □

**Lemma 6.3.2.** *Let $F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$ be part of a minimal resolution of $M$. Assume that the minimal resolution of $G_{\mathfrak{m}}(M)$ has the form*

$$\cdots \to S^a(-s) \xrightarrow{\psi} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \to 0,$$

*that is, assume that the first shift is pure. Set $N = \operatorname{Im}(\phi_1)$ and $\mathcal{F} = \{N_i = \mathfrak{m}^i F_0 \cap N\}_{i\in\mathbb{Z}}$ ($\mathfrak{m}^i = S$ for $i \leq 0$). Then*
*(i) $N_i = \mathfrak{m}^{i-s}N$ for all $i \geq 0$.*
*(ii) $\operatorname{rank}(F_1) = a$.*
*(iii) $v(\phi_1) = s$*
*(iv) The sequence*

$$G_{\mathfrak{m}}(F_1)(-s) \xrightarrow{\text{in}(\phi_1)} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \to 0$$

*can be extended to a minimal resolution of $G_{\mathfrak{m}}(M)$.*
*(v) $\operatorname{Im}(\text{in}(\phi_1)) \cong G_{\mathfrak{m}}(N)(-s)$.*

*Proof.* (i) By the Artin-Rees lemma, we know that $\mathcal{F}$ is an $\mathfrak{m}$-stable filtration. Consider the module

$$\begin{aligned}
\frac{\mathfrak{m}^i F_0/\mathfrak{m}^{i+1}F_0}{(\mathfrak{m}^i F_0 \cap N)/(\mathfrak{m}^{i+1}F_0 \cap N)} &\cong \frac{\mathfrak{m}^i F_0/\mathfrak{m}^{i+1}F_0}{(\mathfrak{m}^i F_0 \cap N + \mathfrak{m}^{i+1}F_0)/\mathfrak{m}^{i+1}F_0} \\
&\cong \frac{\mathfrak{m}^i F_0}{\mathfrak{m}^i F_0 \cap N + \mathfrak{m}^{i+1}F_0} \\
&\cong \frac{\mathfrak{m}^i F_0/(\mathfrak{m}^i F_0 \cap N)}{(\mathfrak{m}^i F_0 \cap N + \mathfrak{m}^{i+1}F_0)/(\mathfrak{m}^i F_0 \cap N)} \\
&\cong \frac{(\mathfrak{m}^i F_0 + N)/N}{\mathfrak{m}^{i+1}F_0/(\mathfrak{m}^{i+1}F_0 \cap (\mathfrak{m}^i F_0 \cap N))} \\
&\cong \frac{(\mathfrak{m}^i F_0 + N)/N}{\mathfrak{m}^{i+1}F_0/(\mathfrak{m}^{i+1}F_0 \cap N)} \qquad (6.2) \\
&\cong \frac{(\mathfrak{m}^i F_0 + N)/N}{(\mathfrak{m}^{i+1}F_0 + N)/N} \\
&\cong \frac{\mathfrak{m}^i F_0 + N}{\mathfrak{m}^{i+1}F_0 + N} \\
&\cong \frac{\mathfrak{m}^i (F_0/N)}{\mathfrak{m}^{i+1}(F_0/N)} \\
&\cong \frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1}M}.
\end{aligned}$$

Since all the isomorphisms in the above simplification are natural, we have the exact sequence

$$0 \to G_{\mathcal{F}}(N) \to G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \to 0.$$

By the hypothesis above, it follows that $G_{\mathcal{F}}$ is generated in degree $s$. So we have $N_i = N$ for $i \leq s$ and

$$\frac{N_{s+j}}{N_{s+j+1}} = \mathfrak{m}^j \frac{N_s}{N_{s+1}} \Rightarrow N_{s+j} = \mathfrak{m}^j N_s + N_{s+j+1} = \mathfrak{m}^j N + N_{s+j+1},$$

for $j \geq 1$. As $\mathcal{F}$ is $\mathfrak{m}$-stable, there exists $j_0$ such that $N_{s+j+1} = \mathfrak{m}N_{s+j}$ for all $j \geq j_0$. For $j \geq j_0$, $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1} = \mathfrak{m}^j N + \mathfrak{m}N_{s+j}$. By Nakayama Lemma, $N_{s+j} = \mathfrak{m}^j N$.
We show by descending induction that $N_{s+j} = \mathfrak{m}^j N$ for all $j \leq j_0$. This is true for $j = j_0$ by the previous argument. Assume $N_{s+j+1} = \mathfrak{m}^{j+1} N$ for some $j \leq j_0 - 1$. Then,

$$\mathfrak{m}^{j+1} N \subset \mathfrak{m}N_{s+j} \subset N_{s+j+1} = \mathfrak{m}^{j+1} N.$$

Hence, $N_{s+j+1} = \mathfrak{m}N_{s+j}$ and $N_{s+j} = \mathfrak{m}^j N + \mathfrak{m}N_{s+j}$. By Nakayama Lemma, $N_{s+j} = \mathfrak{m}^j N$.
(ii) By (i), $G_{\mathcal{F}}(N) = G_{\mathfrak{m}}(N)(-s)$. Hence, the map $\psi$ maps the basis elements of $S^a$ to a minimal generating set of $G_{\mathfrak{m}}(N)$. As observed previously, a minimal generating set of $G_{\mathfrak{m}}(N)$ has the same cardinality as a minimal generating set of $N$. The cardinality of the minimal generating set of $N = \mathrm{Im}(\phi_1)$ is $\mathrm{rank}(F_1)$. Hence, $a = \mathrm{rank}(F_1)$.
(iii) Set $r = v(\phi_1)$. By Lemma 6.3.1 and the discussion preceding it, we have a complex

$$G_{\mathfrak{m}}(F_1)(-r) \xrightarrow{\mathrm{in}(\phi_1)} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \to 0.$$

So $\ker(\epsilon)$ contains an element of degree $r$, which forces that $s \leq r$ by (i). Further, note that $N = \mathrm{Im}(\phi_1) \subset \mathfrak{m}^{v(\phi_1)} F_0 = \mathfrak{m}^r F_0$. Hence, $N_j = N$ for $j \leq r$, which forces that $s \geq r$.
(iv) Consider the following sequence

$$\mathfrak{m}^{i-s} F_1 \xrightarrow{\gamma_{i-s}} \mathfrak{m}^i F_0 \xrightarrow{\epsilon_i} \mathfrak{m}^i M \to 0,$$

for $i \geq 0$, where $\gamma_{i-s}$ and $\epsilon_i$ are the restrictions of $\phi_1$ and $\phi_0$ to $\mathfrak{m}^{i-s} F_1$ and $\mathfrak{m}^i F_0$ respectively. Observe that $\ker(\epsilon_i) = N \cap \mathfrak{m}^i F_0 = N_i = \mathfrak{m}^{i-s} N$ and the map $\gamma_{i-s}$ naturally maps $\mathfrak{m}^{i-s} F_1$ surjectively to $\mathfrak{m}^{i-s} N$. Hence, the above sequence is exact. We tensor this exact sequence with $A/\mathfrak{m}$ to get the exact sequence

$$\frac{\mathfrak{m}^{i-s} F_1}{\mathfrak{m}^{i-s+1} F_1} \xrightarrow{\overline{\gamma_{i-s}}} \frac{\mathfrak{m}^i F_0}{\mathfrak{m}^{i+1} F_0} \xrightarrow{\overline{\epsilon_i}} \frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1} M} \to 0,$$

for all $i \geq 0$. Thus, we have an exact sequence

$$G_{\mathfrak{m}}(F_1)(-s) \xrightarrow{\overline{\phi_1}} G_{\mathfrak{m}}(F_0) \xrightarrow{\overline{\phi_0}} G_{\mathfrak{m}}(M) \to 0.$$

By definition, $\overline{\phi_0} = \epsilon$. If one thinks of $\phi_1$, and hence $\gamma_{i-s}$, as a matrix, then it follows that the map $\overline{\gamma_{i-s}}$ can be represented by the matrix corresponding to $\mathrm{in}(\phi_1)$ (note that $s = v(\phi_1)$ from (iii)). Therefore, $\overline{\phi_1} = \mathrm{in}(\phi_1)$. Further, as $G_{\mathfrak{m}}(F_1) = S^a$ from (ii), we have that $\mathrm{in}(\phi_1)$ maps minimally onto $\ker(\epsilon)$ and hence, the above short exact sequence can be extended to a minimal resolution of $G_{\mathfrak{m}}(M)$.
(v) This follows from the exact sequences in the arguments presented for (i) and (iv) above. □

**Theorem 6.3.3.** *Let $M$ be a finitely generated $A$-module. Assume that $G_{\mathfrak{m}}(M)$ has a pure resolution. Let $\mathbb{F}$ be a minimal free resolution of $M$. Then $\mathrm{in}(\mathbb{F})$ is a minimal free resolution of $G_{\mathfrak{m}}(M)$.*

*Proof.* We proceed by induction on $\mathrm{pdim}(M)$. For $\mathrm{pdim}(M) = 0$, $M$ is free and so is $G_{\mathfrak{m}}(M)$. Thus, the statement of the theorem holds.
Suppose $\mathrm{pdim}(M) = p \geq 1$. Suppose the free resolution of $M$ is

$$0 \to F_p \xrightarrow{\phi_p} \cdots \to F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0,$$

and let $c_i = v(\phi_i)$ and $d_i = \sum_{j=1}^{i} c_j$. Let $N = \mathrm{Im}(\phi_1)$. Since $M$ has a pure resolution, by Lemma 6.3.2,

$$G_{\mathfrak{m}}(F_1)(-d_1) \xrightarrow{\mathrm{in}(\phi_1)} G_{\mathfrak{m}}(F_0) \to M \to 0$$

can be extended to a minimal resolution of $G_{\mathfrak{m}}(M)$, and $\mathrm{Im}(\mathrm{in}(\phi_1)) = G_{\mathfrak{m}}(N)(-d_1)$. Hence, $G_{\mathfrak{m}}(N)$ has a pure resolution as well. Since $\mathrm{pdim}(N) = p - 1$, by the induction hypothesis,

$$0 \to G_{\mathfrak{m}}(F_p)(-d_p + d_1) \xrightarrow{\mathrm{in}(\phi_p)} \cdots \to G_{\mathfrak{m}}(F_1)(-d_1 + d_1) \xrightarrow{\mathrm{in}(\phi_1)} G_{\mathfrak{m}}(N) \to 0$$

is a minimal resolution of $G_{\mathfrak{m}}(N)$. It follows that

$$0 \to G_{\mathfrak{m}}(F_p)(-d_p) \xrightarrow{\mathrm{in}(\phi_p)} \cdots \to G_{\mathfrak{m}}(F_1)(-d_1) \xrightarrow{\mathrm{in}(\phi_1)} G_{\mathfrak{m}}(F_0) \to M \to 0,$$

which is precisely $\mathrm{in}(\mathbb{F})$, is a minimal resolution of $M$. $\square$

To prove the next major result, we require the following technical lemma.

**Lemma 6.3.4.** *Let $R$ be a Noetherian local ring and $N$ be a Cohen-Macaulay $R$-module. Suppose $K$ is a non-zero submodule of $N$. Then, $\dim(K) = \dim(N)$.*

*Proof.* The proof follows by induction on $\dim(N)$. The statement is trivial for $\dim(N) = 0$.
Assume that the statement of the lemma holds for $\dim(N) < d$. Suppose $\dim(N) = d \geq 1$. Let $K$ be a non-zero submodule of $N$. There exists $x \in R$ such that $x$ is a nonzerodivisor on $N$, and $K \not\subset xN$ (by Krull's intersection theorem). Then, $N/xN$ is a Cohen-Macaulay $R$-module of dimension $d-1$. Note that $K/(K \cap xN)$ naturally injects properly into $N/xN$ and hence, by the induction hypothesis, $\dim(K/(K \cap xN)) = \dim(N/xN) = d - 1$.
Observe that since $xK \subset K \cap xN$, $K/xK$ naturally maps onto $K/(K \cap xN)$. Hence, $\dim(K/xK) \geq d-1$. The element $x$ is also a nonzerodivisor on $K$, which forces $\dim(K/xK) = \dim(K) - 1$. Hence, $\dim(K) \geq d$, and further, $\dim(K) = d$ as $K$ is a submodule of $N$. $\square$

**Theorem 6.3.5.** *Let $M$ be a Cohen-Macaulay $A$-module and let $p = \mathrm{pdim}(M)$. Let $\beta_i = \beta_i(M)$ and*

$$\mathbb{F} : 0 \to F_p \xrightarrow{\phi_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

*be a minimal resolution of $M$. Let $c_i = v(\phi_i)$ and $d_i = \sum_{j=1}^{i} c_j$. The following conditions are equivalent:*
*(i) $G_{\mathfrak{m}}(M)$ has a pure resolution.*
*(ii) The following hold:*
*(a)* $\mathrm{in}(\mathbb{F})$ *is acyclic.*
*(b) $\beta_i = b_i\beta_0$ for $i = 1, \ldots, p$, where $b_i = (-1)^{i-1}\prod_{j\neq i}\frac{d_j}{d_j - d_i}$.*
*(c) The multiplicity of $M$,*

$$e_0(M) = \frac{\beta_0}{p!}\prod_{i=1}^{p} d_i.$$

*Proof.* Suppose $G_{\mathfrak{m}}(M)$ has a pure resolution. By Theorem 6.3.3, $\mathrm{in}(\mathbb{F})$ is a minimal resolution of $G_{\mathfrak{m}}(M)$ and hence, $\beta_i(G_{\mathfrak{m}}(M)) = \beta_i$ and the shifts of the minimal resolution of $G_{\mathfrak{m}}(M)$ are $d_1, \ldots, d_p$. Recall that $\dim(M) = \dim(G_{\mathfrak{m}}(M))$ ([5], Theorem 4.5.6). By the Auslander-Buchsbaum formula, $\mathrm{depth}(G_{\mathfrak{m}}(M)) = n - \mathrm{pdim}(G_{\mathfrak{m}}(M)) = n - \mathrm{pdim}(M) = \mathrm{depth}(M)$. Hence, $G_{\mathfrak{m}}(M)$ is Cohen-Macaulay. The statements (ii)(b) and (ii)(c) thus follow from Theorem 6.1.2 and [4] (page 88).
Conversely, if $\mathrm{in}(\mathbb{F})$ is acyclic and the Betti numbers satisfy the Herzog-Kühl conditions, then by Theorem 6.1.2, $E = \mathrm{coker}(\mathrm{in}(\phi_1))$ is Cohen-Macaulay of dimension $n - p$ (by the Auslander-Buchsbaum formula). Recall that we also have a surjective homomorphism $\epsilon : G_{\mathfrak{m}}(F_0) \to G_{\mathfrak{m}}(M)$ with $\epsilon \circ \mathrm{in}(\phi_1) = 0$. Therefore, $\mathrm{Im}(\mathrm{in}(\phi_1)) \subset \ker(\epsilon)$ and we have an exact sequence

$$0 \to K \to E \to G_{\mathfrak{m}}(M) \to 0.$$

Note that $\dim(K) \leq \dim(E) = \dim(G_{\mathfrak{m}}(M))$. As multiplicity of $E$ equals multiplicity of $G_{\mathfrak{m}}(M)$, the degree of the Hilbert polynomial of $K$ must be smaller than the degree of the Hilbert polynomial of $E$ and hence, $\dim(K) < \dim(E)$. By Lemma 6.3.4, $K = 0$. So $G_{\mathfrak{m}}(M) \cong E$ has a pure resolution. □

# Chapter 7

# Explicit construction of some resolutions

## 7.1 The Taylor resolution

Let $S = \mathsf{k}[x_1, \ldots, x_n]$ and $f_1, \ldots, f_t$ be non-constant monomials in $S$.
Let $F_s$ be the free module on the basis elements $\{v_I : I \subset \{1, \ldots, t\}, |I| = s\}$. Set $f_I = \mathrm{LCM}(f_i : i \in I)$. Let $F_\phi = 1$.
Suppose $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, t\}$ and $J \subset \{1, \ldots, t\}$, $|J| = s - 1$. Then, define

$$c_{IJ} = \begin{cases} 0 & J \not\subset I \\ (-1)^k f_I/f_J & if\ I = J \cup \{i_k\}\ for\ some\ k. \end{cases}$$

Define $d_s : F_s \to F_{s-1}$ as $d_s(v_I) = \sum_J c_{IJ} v_J$. Finally, define the Taylor's complex $T(f_1, \ldots, f_t)$ to be

$$0 \to F_t \xrightarrow{d_t} F_{t-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to 0.$$

We prove that the Taylor's complex is a resolution of $S/\langle f_1, \ldots, f_t\rangle$ by induction on $t$. The base case $t = 1$ is clear. Assume that $T(h_1, \ldots, h_t)$ is a resolution of $S/(h_1, \ldots, h_t)$ for any monomials $h_1, \ldots, h_t \in S$.
Consider monomials $f_1, \ldots, f_{t+1}$ in $S$. For $i = 1, \ldots, t$, let $g_i = f_i/\mathrm{GCD}(f_i, f_{t+1})$. By the induction hypothesis, $T(g_1, \ldots, g_t)$ is a resolution of $S/(g_1, \ldots, g_t)$. Consider the short exact sequence

$$0 \to S/(g_1, \ldots, g_t) \xrightarrow{-f_{t+1}} S/(f_1, \ldots, f_t) \to S/(f_1, \ldots, f_{t+1}) \to 0.$$

The first map in this short exact sequence can be induced to a map of complexes

$$
\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
G_t & \xrightarrow{\varphi_t} & F_t & & \\
\downarrow{\scriptstyle \delta_t} & & \downarrow{\scriptstyle d_t} & & \\
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
G_2 & \xrightarrow{\varphi_2} & F_2 & & \\
\downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle d_2} & & \\
G_1 & \xrightarrow{\varphi_1} & F_1 & & \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle d_1} & & \\
G_0 & \xrightarrow{\varphi_0} & F_0 & & \\
\downarrow & & \downarrow & & \\
S/(g_1,\dots,g_t) & \xrightarrow{f_{t+1}} & S/(f_1,\dots,f_t) & \longrightarrow & S/(f_1,\dots,f_{t+1})
\end{array}
$$

where $G_s$ is a free module on the basis elements $\{w_I : I \subset \{1,\dots,t\}, |I| = s\}$, $\delta_s$ is the usual map in the Taylor complex and $\varphi_s(w_I) = -(g_I/f_I)f_{t+1}v_I$ for $s \in \{1,\dots,t\}$.

**Claim:** $d_i\phi_i = \phi_{i-1}\delta_i$ for all $i \geq 1$.

*Proof.* Let $I \subset S$, $|I| = i$ and $\Lambda = \{J \subset \{1,\dots,t\} : |J| = i-1\}$ Then,

$$d_i\phi_i(w_I) = d_i(-(g_I/f_I)f_{t+1}v_I) = -(g_I/f_I)f_{t+1}\sum_{J\in\Lambda} c_{IJ}v_J,$$

where

$$c_{IJ} = \begin{cases} 0 & J \not\subset I \\ (-1)^k f_I/f_J & if\ I = J \cup \{i_k\}\ for\ some\ k. \end{cases}$$

On the other hand,

$$\phi_{i-1}\delta_i(w_I) = \phi_{i-1}(\sum_{J\in\Lambda} b_{IJ}w_J) = -f_{t+1}\sum_{J\in\Lambda} b_{IJ}(g_J/f_J)v_J,$$

where

$$b_{IJ} = \begin{cases} 0 & J \not\subset I \\ (-1)^k g_I/g_J & if\ I = J \cup \{i_k\}\ for\ some\ k. \end{cases}$$

In both cases, the coefficient of $v_J$ if $J \cup i_k = I$ is precisely $(-1)^k g_I/f_J$. □

Note that $\deg((g_I/f_I)f_{t+1}) = \deg(\mathrm{LCM}(f_I, f_{t+1})/f_I) \geq 1$, which implies that all the elements of the matrix corresponding to $\phi_i$ belong to the homogeneous maximal ideal $\langle x_1, \ldots, x_n \rangle$. Hence, using the mapping cone theorem, we have a resolution of $S/(f_1, \ldots, f_{t+1})$

$$H_\bullet : 0 \to H_{t+1} \xrightarrow{\epsilon_{t+1}} H_t \to \cdots \to H_1 \xrightarrow{\epsilon_1} H_0 \to 0,$$

where $H_0 = F_0$ and $H_i = G_{i-1} \oplus F_i$ for $i \geq 1$. The maps $\epsilon_i : H_i \to H_{i-1}$ are defined as

$$\epsilon_i(g, f) = (\delta_{i-1}(g), -d_i(f) - \phi_{i-1}(g)).$$

Observe that $H_i$ is a free module on the basis elements $\{w_I : I \subset \{1, \ldots, t\}, |I| = i-1\} \cup \{v_I : I \subset \{1, \ldots, t\}, |I| = i\}$. Consider the following basis of $H_i$: $\{u_I : I \subset \{1, \ldots, t+1\}, |I| = i\}$, where

$$u_I = \begin{cases} w_{I-\{t+1\}} & t+1 \in I \\ (-1)^{|I|+1} v_I & t+1 \notin I. \end{cases}$$

Suppose $t+1 \in I$ and $|I| = i$. Let $\Lambda_i = \{J \subset \{1, \ldots, t\} : |J| = i\}$. Then,

$$\begin{aligned} \epsilon_i(u_I) &= \epsilon_i(w_{I-\{t+1\}}) \\ &= (\delta_{i-1}(w_{I-\{t+1\}}), -\phi_{i-1}(w_{I-\{t+1\}})) \\ &= (\sum_{J \in \Lambda_{i-2}} b_{IJ} w_J, (g_{I-\{t+1\}}/f_{I-\{t+1\}}) f_{t+1} v_{I-\{t+1\}}) \\ &= (\sum_{J \in \Lambda_{i-2}} b_{IJ} u_J, (-1)^{i+1} (g_{I-\{t+1\}}/f_{I-\{t+1\}}) f_{t+1} u_{I-\{t+1\}}) \qquad (7.1) \\ &= (\sum_{J \in \Lambda_{i-2}} b_{IJ} u_J, a_{IJ} u_{I-\{t+1\}}) \\ &= \sum_{J \in \Lambda_{i-1}} a_{IJ} u_J. \end{aligned}$$

In the above computation, the following simplification has been used: for $I \subset \{1, \ldots, t\}$,

$$\begin{aligned} g_I f_{t+1} &= \mathrm{LCM}(\frac{f_i}{\mathrm{GCD}(f_i, f_{t+1})} : i \in I) f_{t+1} \\ &= \mathrm{LCM}(\frac{LCM(f_i, f_{t+1})}{f_{t+1}} : i \in I) f_{t+1} \qquad (7.2) \\ &= \mathrm{LCM}(f_i : i \in I \cup \{t+1\}). \end{aligned}$$

Suppose $t+1 \notin I$ and $|I| = i$. Then,

$$\begin{aligned} \epsilon_i(u_I) &= \epsilon_i((-1)^{|I|+1} v_I) \\ &= (0, -(-1)^{|I|+1} d_i(v_I)) \\ &= (0, (-1)^{|I|} \sum_{J \in \Lambda_i} c_{IJ} v_J) \qquad (7.3) \\ &= \sum_{J \in \Lambda_{i-1}} a_{IJ} u_J. \end{aligned}$$

The differentials maps $\epsilon$ are hence precisely the maps in the Taylor complex whose free modules are generated by $u_I$'s. Hence, we have proved by induction that the Taylor complex provides a resolution of monomial ideals.
The Taylor's resolution need not be minimal. For example, suppose $S = \mathsf{k}[x_1, x_2, x_3]$, $f_1 = x_1x_2$, $f_2 = x_2x_3$, $f_3 = x_1x_3$. Let $I = \{1, 2, 3\}$ and $J = \{1, 2\}$. Then, $f_I = f_J = x_1x_2x_3$ and $c_{IJ} = (-1)^3 f_I/f_J = -1 \notin \langle x_1, x_2, x_3\rangle$.
However, if $I$ is a stable ideal, we can construct a minimal free resolution of $I$.

## 7.2 The Eliahou-Kervaire resolution

Let $S = \mathsf{k}[x_1, \ldots, x_n]$.
Suppose $I$ is a monomial ideal in $S$. We denote by $G(I)$ a minimal generating set of $I$. Given a non-constant monomial $a = x_1^{a_1} \ldots x_n^{a_n}$, let $\max(a) = \max\{i : a_i > 0\}$ and $\min(a) = \min\{i : a_i > 0\}$. Define $\min(1) = \infty$.
Recall that a monomial ideal $I$ is said to be stable if for every monomial $w \in I$, $x_i w/x_{max(w)} \in I$ for all $i < \max(w)$. We begin with some lemmas on stable ideals which shall be needed in the construction of the Eliahou-Kervaire resolution.

**Lemma 7.2.1.** *Let $I$ be a stable monomial ideal with canonical generating set $G(I)$. For every monomial $w \in I$, there is a unique decomposition*

$$w = u.y$$

*with $u \in G(I)$ and* $\max(u) \leq \min(y)$.

*Proof.* Given a monomial $w \in I$, there exists $v \in G(I)$ and $z \in S$ such that $w = v.z$.
Suppose $\max(v) > \min(z)$. Let $i = \min(z)$ and $m = \max(v)$. Then, by the stability hypothesis, $x_i v/x_m \in I$ and hence we can write

$$w = (x_i v/x_m).(x_m z/x_i),$$

where $x_i v/x_m$ is itself a multiple of some monomial $v' \in I$. Hence, $w = v'.z'$ for some suitable monomial $z' \in S$.
Note that on passage from $v = x_1^{b_1} \ldots x_n^{b_n}$ to $v'$, the non-negatively valued function $f(v) = \sum_{i=1}^{n}$ is strictly decreasing. Hence, after finitely many iterations of the above process, we must have $w = u.y$, where $u \in G(I)$ and $\max(u) \leq \min(y)$.
Suppose $w = u.y = u'.y'$, where $u, u' \in G(I)$, $\max(u) \leq \min(y)$ and $\max(u') \leq \min(y')$, then $u, u'$ are both initial segments of $u$ and one of them must divide the other. Since $u, u' \in G(I)$, this forces that $u = u'$ and hence, $y = y'$. Thus, the decomposition is unique. ☐

This unique decomposition of a monomial $w \in I$ will be called the **canonical $I$-decomposition** of $w$.

For a stable ideal $I$, let $M(I)$ denote the set of all monomials in $I$. Define the **decomposition function**

$$g : M(I) \to G(I)$$

by $g(w) = u$ if $w = u.y$ is the unique $I$-canonical decomposition of $w$.

**Lemma 7.2.2.** *Let $I$ be a stable ideal and let $g : M(I) \to G(I)$ be its decomposition function. Then for all $w \in M(I)$, and all monomials $y$, the equation $g(wy) = g(w)$ holds iff $\max(g(w)) \le \min(y)$.*

*Proof.* Suppose $g(wy) = g(w)$. Then the canonical decomposition of $wy$ reads

$$wy = g(wy).z = g(w).z,$$

where $\max(g(w)) \le \min(z)$. Since $g(w)$ divides $w$, $y$ must divide $z$, which forces that $\min(z) \le \min(y)$. Hence, $\max(g(w) \le \min(y)$.
Suppose $\max(g(w)) \le \min(y)$. Suppose the canonical decomposition of $w$ is $w = g(w).z$, $\max(g(w)) \le \min(z)$. Then $wy = g(w).yz$ is the canonical decomposition of $wy$, since $\max(g(w)) \le \min(yz)$. Therefore, $g(wy) = g(w)$. □

**Lemma 7.2.3.** *Let $I$ be a stable monomial ideal with decomposition function $g : M(I) \to G(I)$. Then, for any monomial $a$ and any $w \in M(I)$,*
*(i) $g(ag(w)) = g(aw)$,*
*(ii) $\max(g(aw)) \le \max(g(w))$.*

*Proof.* (i) Assume first that $a = x_i$.
*Case 1*: If $i \ge \max(g(w))$, then $g(x_i w) = g(w)$ by Lemma 7.2.2 and $g(w).x_i$ is itself a canonical decomposition of $g(w)x_i$, which implies that $g(x_i g(w)) = g(w)$.
*Case 2*: If $i < \max(g(w))$, then $g(x_i w) = g(x_i g(w) y)$ for some monomial $y \in S$ with $\max(g(w)) \le \min(y)$. Note that $\max(x_i g(w)) = \max(g(w)) \le \min(y)$. Hence, by Lemma 7.2.2, $g(x_i w) = g(x_i g(w)$.
For an arbitrary monomial $a$, the proof follows by induction on the degree of $a$. For example, $g(x_i x_j w) = g(x_i g(x_j w)) = g(x_i x_j g(w))$.
(ii) As in (i), it suffices to prove the statement for $a = x_i$, as the rest of the proof follows by induction on degree.
If $i \ge \max(w)$, then $g(x_i w) = g(w)$ by Lemma 7.2.2.
If $i < \max(w)$, then

$$\max(g(x_i g(w))) \le \max(x_i g(w)) \le \max(g(w)),$$

and since $g(x_i w) = g(x_i g(w))$ by (i), we have $\max(g(x_i w)) \le \max(g(w))$. □

**Lemma 7.2.4.** *Let $w \in M(I)$ be a monomial in $I$ and let $a$ be a monomial in $S$. Then*

$$g(a.w) \le g(w)$$

*in the graded reverse lexicographic order.*

*Proof.* Again, it suffices to prove the statement for $a = x_i$.
If $\max(g(w)) \le i$, then $g(x_iw) = g(w)$ by Lemma 7.2.2, and the statement follows trivially.
If $\max(g(w)) > i$, let

$$x_ig(w) = g(x_ig(w)).y = g(x_iw).y$$

with $\max(g(x_iw)) \le \min(y)$ be the canonical decomposition of $x_ig(w)$ (the second equality is by Lemma 7.2.3).
We must have $\deg(y) > 0$, since $\deg(y) = 0$ would imply that $g(x_iw) \in G(I)$ is a proper multiple of $g(w) \in G(I)$. Hence, $\deg(g(x_iw)) \le \deg(g(w))$.
If $\deg(g(x_iw)) < \deg(g(w))$, $g(x_iw) < g(w)$ in the graded reverse lexicographic order. If $\deg(g(x_iw)) = \deg(g(w))$, then $\deg(y) = 1$, that is, $y$ is a variable, say, $x_j$.
Since $i < \max(g(w))$, and $\max(g(x_iw)) \le j$, it follows that $j = \max(g(w))$. The equation $x_ig(w) = g(x_iw)x_j$ forces that the exponent of $x_j$ in $g(w)$ is strictly larger than the exponent of $x_j$ in $g(x_iw)$.
Since $\max(g(x_iw)) \le j$, $g(x_iw) < g(w)$ as desired. □

We now proceed to describe the minimal graded free resolution $(L_*(I), d)$ of an arbitrary stable monomial ideal $I \subset S$.
Define a symbol $e(i_1, \dots, i_q; u)$ to be **admissible** if the following three conditions are satisfied:
(i) $u \in G(I)$.
(ii) $i_1, \dots, i_q$ are integers such that $1 \le i_1 < \cdots < i_q \le n$.
(iii) $i_q < m = \max(u)$.
In this definition, $q$ may be 0.
Let $L_q = L_q(I)$ be the free $S$-module on the set of all admissible symbols $e(i_1, \dots, i_q; u)$ for fixed $q \ge 0$. In particular, $L_0(I)$ is the free $S$-module with set of generators $e(u)$ for $u \in G(I)$.
We define the map of $S$-modules

$$\alpha : L_0 \to I$$

by $\alpha(e(u)) = u$.
In order to define $d : L_q \to L_{q-1}$ for $q \ge 1$, we need some more notations as follows: Let $e(i_1, \dots, i_q; u)$ be an admissible symbol. Denote by $\sigma$ the sequence $(i_1, \dots, i_q)$. If $\sigma = (i_1, \dots, i_q)$, we denote by $\sigma_r$ the sequence $\sigma_r = (i_1, \dots, \hat{i}_r, \dots, i_q)$ in which $i_r$ has been deleted.
Let $u_r = g(x_{i_r}u)$ and $y_r = x_{i_r}u/u_r$. Then, by definition, $\max(u_r) \le \min(y_r)$. We write $m_r = \max(u_r)$ and denote by $A(\sigma; u) \subset \{1, \dots, q\}$ the set of values of $r$ for which $\max(i_1, \dots, \hat{i}_r, \dots, i_q) < m_r$, or equivalently, the set of values of $r$ for which $e(\sigma_r; u_r)$ is an admissible symbol.
The map $d : L_q \to L_{q-1}$ is the $S$-module map determined by

$$de(\sigma; u) = \sum_{r=1}^{q} (-1)^r x_{i_r} e(\sigma_r; u) - \sum_{r \in A(\sigma;u)} (-1)^r y_r e(\sigma_r; u_r).$$

**Remark 7.2.5.** Observe that since $x_{i_r}$ is not a minimal generator of $I$, it follows that $\deg(y_r) \ge 1$ and thus, $d(L_q) \subset \langle x_1, \dots, x_n \rangle L_{q-1}$.

The module $L_q(I)$ is endowed with a natural multigrading defined by

$$\deg(z.e(i_1,\dots,i_q;u)) = zx_{i_1}\dots x_{i_q}u,$$

for $q \geq 1$ and $\deg(z.e(u)) = zu$. Note that the maps $d : L_q \to L_{q-1}$, as well as $\alpha : L_0 \to I$, preserve the multigrading.

**Theorem 7.2.6.** *$(L_*(I), d)$ as described above is a minimal free graded resolution of $I$ over $S$.*

**Proposition 7.2.7.** *$(L_*(I), d)$ as described above is a complex.*

*Proof.* We first need to check that $(L_*(I), d)$ is a complex. To do so, we will exhibit $(L_*(I), d)$ as the quotient of another complex.
Let $C_q$ be the free $R$-module on all symbols $e(i_1,\dots,i_q)$ satisfying only the two conditions
(i) $u \in G(I)$.
(ii) $i_1,\dots,i_q$ are integers such that $1 \leq i_1 < \dots < i_q \leq n$.
Define $D : C_q \to C_{q-1}$ to be the $R$-module map determined by

$$De(\sigma;u) = \sum_{r=1}^{q}(-1)^r x_{i_r}e(\sigma_r;u) - \sum_{r=1}^{q}(-1)^r y_r e(\sigma_r;u_r),$$

where, as before, $u_r = g(x_{i_r}u)$ and $y_r = x_{i_r}u/u_r$.
To prove that $(C_*, D)$ is a complex, it is convenient to cut the operator $D$ in two: Let $D = D_1 - D_2$, where

$$D_1 e(\sigma_u) = \sum_{r=1}^{q}(-1)^r x_{i_r}e(\sigma_r;u),$$

$$D_2 = \sum_{r=1}^{q}(-1)^r y_r e(\sigma_r;u_r).$$

Let $\sigma = \{i_1,\dots,i_q\}$, $\sigma_r = \{i_1,\dots,\hat{i}_r,\dots,i_q\} = \{j_1,\dots,j_{q-1}\}$ and $\sigma_{r,s} = \sigma_{s,r} = \{i_1,\dots,\hat{i}_r,\dots,\hat{i}_s,\dots,i_q\}$ for $r < s$. Let $u'_s = g(x_{j_s}u)$ and $y'_s = x_{j_s}u/u'_s$ for $s = 1,\dots,q-1$.

$$\begin{aligned} D_1^2(e(\sigma;u)) &= \sum_{r=1}^{q}(-1)^r x_{i_r}D_1(e(\sigma_r;u)) \\ &= \sum_{r=1}^{q}(-1)^r x_{i_r}\sum_{s=1}^{q-1}(-1)^s x_{j_s}e((\sigma_r)_s;u) \qquad (7.4) \\ &= \sum_{r=1}^{q}\sum_{s=1}^{q-1}(-1)^r x_{i_r}x_{j_s}e((\sigma_r)_s;u). \end{aligned}$$

For $k < t$, the basis element $e(\sigma_{kt};u)$ appears in two summands in the above summation: once when $r = k$,$s = t-1$ with coefficient $(-1)^{k+t-1}x_{i_k}x_{i_t}$ and once when $r = t$,$s = k$ with coefficient $(-1)^{k+t}x_{i_k}x_{i_t}$. Hence, $D_1^2 = 0$.

$$(D_2D_1 + D_1D_2)(e(\sigma; u)) = \sum_{r=1}^{q}\sum_{s=1}^{q-1}(-1)^r x_{i_r}(-1)^s y'_s e((\sigma_r)_s; u'_s) + \sum_{r=1}^{q}\sum_{s=1}^{q-1}(-1)^r y_r(-1)^s x_{j_s} e((\sigma_r)_s; u_r). \tag{7.5}$$

For $k < t$, the basis element $e(\sigma_{kt}; u_r)$ appears in two summands in this summation: once when $r = k$, $s = t-1$ with coefficient $(-1)^{k+t-1}x_{i_k}y'_{t-1} = (-1)^{k+t-1}x_{i_k}y_t$ and once when $r = t$, $s = k$ with coefficient $(-1)^{k+t}y_t x_{i_k}$. Hence, $D_2D_1 + D_1D_2 = 0$.
Finally, in order to calculate $D_2^2(e(\sigma; u))$, let

$$u_{rs} = g(x_{i_r}x_{i_s}u)$$

and let $y_{rs} = x_{i_r}g(x_{i_s}u)/g(x_{i_r}x_{i_s}u)$. By Lemma 7.2.3(i), $u_{rs} = g(x_{i_r}g(x_{i_s})u) = g(x_{i_s}g(x_{i_r})u)$, and thus

$$\begin{aligned} D_2^2 e(\sigma; u) = &\sum_{1\le s<r\le n}(-1)^{r+s}y_r y_{sr} e(\sigma_{sr}; u_{sr}) \\ &+ \sum_{1\le r<s\le n}(-1)^{r+s-1}y_r y_{sr} e(\sigma_{rs}; u_{rs}). \end{aligned} \tag{7.6}$$

Clearly, $y_r y_{sr} = y_s y_{rs}$ and hence, $D_2^2 = 0$. Thus, $D^2 = 0$.
Let $N_q \subset C_q$ be the submodule generated by the symbols $e(i_1, \ldots, i_q; u)$ with $\max(u) \le i_q$. We claim that $N_*$ is a subcomplex of $C_*$, that is, $D(N_q) \subset N_{q-1}$.
Indeed, if $\max(u) \le i_q$, Lemma 7.2.2 forces $u_q = g(x_{i_q}u) = u$ and hence, $y_q = x_{i_q}$. Hence, the last term $(-1)^q y_q e(\sigma_q; u_q)$ in $D_2e(\sigma; u)$ coincides with the last term $(-1)^q x_{i_q} e(\sigma_q; u)$ of $D_1(\sigma; u)$. It follows that if $\max(u) \le i_q$, then

$$De(\sigma; u) = \sum_{r=1}^{q-1}(-1)^r x_{i_r} e(\sigma_r; u) - \sum_{r=1}^{q-1}(-1)^r y_r e(\sigma_r; u_r).$$

Since, by Lemma 7.2.3(ii),

$$\max(u_r) = \max(g(x_{i_r})u) \le \max(g(u)) = \max(u) \le i_q,$$

and $i_q$ is the last index in $\sigma_r$ for $r = 1, \ldots, q-1$, it follows that $De(\sigma; u) \in N_{q-1}$.
Clearly, $L_* = C_*/N_*$ and the boundary operator $d : L_q \to L_{q-1}$ is induced by the boundary operator $D$ on $C_*$. Hence, $d^2 = 0$. The vanishing of the composition $\alpha \circ d$ is easily verifiable by direct computation. Hence, $(L_*(I), d)$ is a complex. □

In order to prove $\ker(d_q) \subset \mathrm{Im}(d_{q+1})$, we define a "normal form" to which every element of $L_q$ may be reduced modulo $\mathrm{Im}(d_{q+1})$ and show that $\ker(d_q)$, respectively $\ker(\alpha)$, contains no normal element except 0.
Let $B$ be the natural $\mathsf{k}$-basis for $L_q$, that is, $B$ contains the elements $z.e(\sigma; u)$ where $z$ is a monomial in $S$ and $e(\sigma; u)$ is an admissible symbol. Elements of $B$ are called **terms**.

**Definition 7.2.8.** *A term $z.e(i_1, \ldots, i_q; u)$ will be called **normal** if $z = 1$, or if $\min(z) \geq i_1$, when $q \geq 1$, or $\min(z) \geq \max(u)$ when $q = 0$.*
*An element $f \in L_q$ is normal if it is a linear combination of normal terms. The element 0 is normal.*

Given two sequences $\sigma = (i_1, \ldots, i_q)$, $\sigma' = (j_1, \ldots, j_q)$ of the same length $q$, define $\sigma < \sigma'$ if $x_{i_1} x_{i_2} \ldots x_{i_q} < x_{j_1} \ldots x_{j_q}$ in the graded reverse lexicographic order. Given two terms $z.e(\sigma; u)$, $z'.e(\sigma'; u')$ in $L_q$, define $z.e(\sigma; u) < z'.e(\sigma'; u')$ if either $u < u'$, or $u = u'$ and $\sigma < \sigma'$, or $e(\sigma; u) = e(\sigma'; u')$ and $z < z'$.

**Lemma 7.2.9.** *Let $a = e(i_0, \ldots, i_q; u)$ be a term in $L_{q+1}$, $q \geq 0$. Then $x_{i_0} e(i_1, \ldots, i_q; u)$ is the biggest term in $d(a)$.*

*Proof.* Let $\sigma = (i_0, i_1, \ldots, i_q)$. We have

$$de(\sigma; u) = \sum_{r=0}^{q} (-1)^{r+1} x_{i_r} e(\sigma_r; u) - \sum_{r \in A(\sigma; u)} (-1)^{r+1} y_r e(\sigma_r; u_r).$$

By Lemma 7.2.2, since $i_r < \max(u)$ for all $r = 0, \ldots, q$, $g(x_{i_r} u) \neq g(u) = u$. By Lemma 7.2.4, $u_r = g(x_{i_r}) u < u$. Thus all terms in the second sum are strictly smaller than $x_{i_0} e(\sigma_0; u)$. Further, since $\sigma_r < \sigma_0$ for all $r \geq 1$, it follows that $x_{i_0} e(\sigma_0; u)$ is indeed the biggest term in $de(\sigma; u)$. □

**Lemma 7.2.10.** *Let $b = ze(i_1, \ldots, i_q; u)$ be a non-normal term in $L_q$, $q \geq 0$. Then $b$ is congruent modulo $\operatorname{Im}(d_{q+1})$ to an element whose terms are all strictly smaller than $b$.*

*Proof.* Let $i = \max(z)$. We have $i < i_1$ (respectively $i < \max(u)$ for $q = 0$), since $b$ is assumed to be non-normal.
Consider the term $a = (z/x_i).e(i, i_1, \ldots, i_q; u) \in L_{q+1}$. By Lemma 7.2.9, $b$ is the biggest term in $d(a)$ (and has coefficient -1). Thus, all terms in $b + d(a)$ are strictly smaller than $b$. □

**Proposition 7.2.11.** *Any element in $L_q$, $q \geq 0$, is congruent to some normal element modulo $\operatorname{Im}(d_{q+1})$.*

*Proof.* Let $f \in L_q$. Suppose $f$ is non-normal. By Lemma 7.2.10, we can replace any non-normal term in $f$ by a combination of strictly smaller terms, not changing the class of $f$ modulo $\operatorname{Im}(d_{q+1})$. When iterated, this process must end up with a normal element in finitely many steps because $d$ preserves the multigrading, and there are only finitely many terms with a given multidegree. □

**Lemma 7.2.12.** *Let $b$ be a normal term in $L_q$, $q \geq 0$. Let $b'$ be any term in $L_q$ and assume that the biggest term in $d(b)$ actually appears among the terms in $d(b')$. Then $b \leq b'$.*

*Proof.* Assume first $q = 0$. Then $b$ and $b'$ have the form $b = y.e(u)$, $b' = z.e(v)$. Since $\alpha(b) = uy$, $\alpha(b') = vz$, the hypotheses amount to $uy = vz$ and $\max(u) \leq \min(y)$, by normality of $b$. Thus, $u.y$ is the canonical decomposition of $vz$, and so $u = g(vz)$. By Lemma 7.2.4, we conclude $u = g(vz) \leq g(v) = v$, and so $b \leq b'$.

Assume now $q \geq 1$. Let $b = y.e(i_1, \ldots, i_q; u)$, $b' = z.e(j_1, \ldots, j_q; v)$. By hypothesis, $c = x_{i_1} ye(i_2, \ldots, i_q; u)$, the biggest term in $d(b)$ according to Lemma 7.2.4, appears as a term in $d(b')$. If it appears as a term of the second kind, that is,

$$x_{i_1} ye(i_2, \ldots, i_q; u) = z_r ze(j_1, \ldots, \hat{j}_r, \ldots, j_q; g(x_{j_r} v)),$$

where $z_r = x_{j_r} v / g(x_{j_r} v)$, then $u = g(x_{j_r} v) < g(v) = v$ by Lemma 7.2.4, and so $b < b'$.
If $c$ is equal to a term in $d(b')$ of the first kind, that is,

$$x_{i_1} ye(i_2, \ldots, i_q; u) = x_{j_r} ze(j_1, \ldots, \hat{j}_r; v),$$

then $u = v$. We compare the sequences: if $r > 1$, then we have $i_s = j_s$ for $r + 1 \leq s \leq q$ and $i_r = j_{r-1} < j_r$. Hence, $(i_1, \ldots, i_q) < (j_1, \ldots, j_q)$ and $b < b'$.
If $r = 1$, we have $(i_2, \ldots, i_q) = (j_2, \ldots, j_q)$ and $x_{i_1} y = x_{j_1} z$. By normality of $b$, we have $i_1 = \min(x_{i_1} y) = \min(x_{j_1} z)$. Hence, $i_1 \leq j_1$. If $i_1 = j_1$, this implies $b = b'$. If $i_1 < j_1$, then $(i_1, \ldots, i_q) < (j_1, \ldots, j_q)$ and $b < b'$. □

**Proposition 7.2.13.** *Let $f$ be a non-zero normal element in $L_q$, $q \geq 0$. Then $d(f)$, respectively $\alpha(f)$, is non-zero.*

*Proof.* Let $b$ be the biggest term in $f$ and $c$ the biggest term in $d(b)$, respectively $\alpha(b)$. By Lemma 7.2.12, $c$ cannot cancel against any other term in $d(f)$, respectively $\alpha(f)$. Hence $d(f)$, respectively $\alpha(f)$, is non-zero. □

This finishes the proof of Theorem 7.2.6.

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